lecture 1, spring 2023

Seifert - van Kampen thm

$X = U \cup V$ $\quad U, V$ open, $\quad U, V, U \cap V$ path connected, base-point $x_0 \in U \cap V$

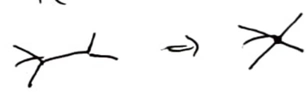
$$\pi_1(V) \xleftarrow{i_2} \pi_1(U \cap V) \xrightarrow{i_1} \pi_1(U) \qquad \pi_1(X) \simeq \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$$

i_1, i_2 - not injective, in general $\qquad \pi_1(U) * \pi_1(V)$ mod relations $i_1(h) = i_2(h)$ $\forall h \in \pi_1(U \cap V)$

X - finite graph. $\qquad$ vertices - edges χ of X

Prop 1) $\pi_1(X) \simeq *\mathbb{Z}^m$, $\quad m = |\text{edges}(X)| - |\text{vertices}| + 1$

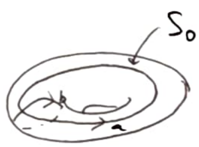
2) X - finite graph, $e \subset X$ edge, not a loop. Then $X \to X/e$ is a homotopy equivalence



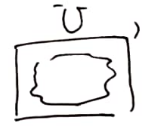
$X \to$ reduce to loops

T^2 torus

1-skeleton



Use MV Theorem



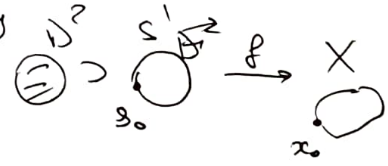
U, $U \sim S_0$ deformation retract

$V = D^2$

$$\pi_1(U) * \pi_1(V)$$
$$aba^{-1}b^{-1} \quad \nwarrow \mathbb{Z} \nearrow \quad \|$$
$$1 \qquad\qquad 1$$

$\Rightarrow \pi_1(T^2) \simeq \underset{a}{\mathbb{Z}} \times \underset{b}{\mathbb{Z}}$

Prop $\quad \pi_1(X \underset{f}{\cup} D^2) \simeq \pi_1(X) / ([f])$ $\quad$ depends only on conj class of $[f]$



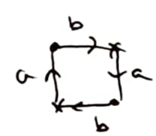
$X \underset{f}{\cup} D^2$

$[f]$

Prop if glue $D^2_1, \dots D^2_n \Rightarrow \pi_1(X) / (f_1, \dots f_n)$

Example Klein bottle, $\mathbb{R}P^2$

$\pi_1(\mathbb{R}P^2) = C_2$



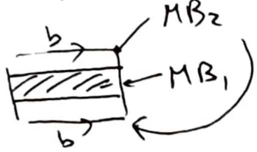
$\langle a b a b^{-1} \rangle$ $\quad G\, b/$ $\quad abab^{-1} = 1$

$abab = b^2 \Rightarrow (ab)^2 = b^2$

$\dfrac{ab = ba^{-1}}{ba = a^{-1}b}$ $\quad \Rightarrow \{a^n b^m\}_{n,m \in \mathbb{Z}}$

$a^{n_1} b^{m_1} a^{n_2} b^{m_2}$

size of $\pi_1(KB)$



MB_2, MB_1

$\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$

$c, d / c^2 = d^2$

How can we distinguish $\mathbb{RP}^2, T^2, KB$?

$\pi_1(KB)$ not commutative $\pi_1(T^2)$ commutative

$G \to [G,G]$ commutator subgroup $[G,G] \triangleleft G$ normal, $G/[G,G]$ max. abelian quotient.

$G = \langle a_i \mid r_j \rangle$ (generators, relations) $\to$ abelianize relations

$H_1(G)$ - first homology group of G

$G = \langle a, b \mid abab^{-1} \rangle \to \underline{a}, \underline{b}$ images in $G/[G,G]$

$\underline{a}+\underline{b}+\underline{a}-\underline{b}=0$

$2\underline{a}=0$

$G/[G,G] \simeq \underset{\underline{b}}{\mathbb{Z}} \oplus \underset{\underline{a}}{\mathbb{Z}/2}$

$\langle a_1 \dots a_m \mid r_1 \dots r_n \rangle$

$n\overset{m}{\left(R \right)}$ relations matrix

$G = \langle a, b \mid a^3 = b^2 \rangle \to 3\underline{a} = 2\underline{b}$

$\underline{a} + 2(\underbrace{\underline{a}-\underline{b}}_{\underline{c}}) = 0$

$H_1(G) = \mathbb{Z}$

abelianized relations

$\langle abab, b^6 \rangle$

$\begin{pmatrix} 2 & 2 \\ 0 & 6 \end{pmatrix}$ row/column manipulations

$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$

<u>Prop</u> $G \longrightarrow H_1(G)$ is a natural construction

$G \xrightarrow{\varphi} K$ homomorphism $H_1(G) \xrightarrow{H_1(\varphi)\ \varphi_*} H_1(K)$

induced hom. of abelian groups

functor Groups $\longrightarrow$ Ab

Cat. of groups — Cat of ab. groups

$\mathcal{C} \xrightarrow{F} \mathcal{D}$ functor

$Ob(\mathcal{C}) \ni X$ $\quad F(X) \in Ob(\mathcal{D})$

$X \xrightarrow{\alpha} Y$ $\quad F(\alpha): F(X) \to F(Y)$

$X \xrightarrow{id} X$ $\quad F(id_X) = id_{F(X)}$

$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ $\quad F(\beta\alpha) = F(\beta) \circ F(\alpha)$

(comp. in $\mathcal{C}$ — comp. in $\mathcal{D}$)

$\mathcal{C}$ cat

Objects $Ob(\mathcal{C})$

$\forall X, Y \in Ob(\mathcal{C})$ $Hom(X,Y) = Hom_{\mathcal{C}}(X,Y)$ morphisms

$Z \xrightarrow{\beta} X \xrightarrow{\alpha} Y$ $\quad id_X$

$\alpha \cdot id_X = \alpha \quad \forall \alpha \in Hom(X,Y)$

$id_X \cdot \beta = \beta \quad \forall \beta \in Hom(Z,X)$

$Hom(X,Y) \times Hom(Y,Z) \to Hom(X,Z)$ assoc

$\alpha \times \beta \longmapsto \beta\alpha$

$\gamma(\beta\alpha) = (\gamma\beta)\alpha$

Sets $\to$ F-vect

$X \longmapsto FX$ vect space basis X.

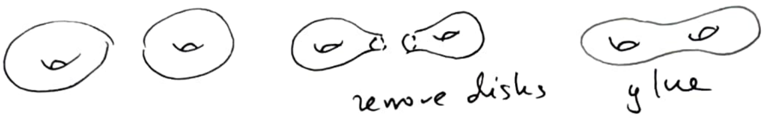
H_1 : Groups $\to$ Ab is a functor

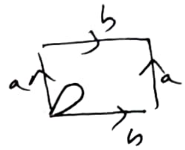
$G \longmapsto G/[G,G]$

$G \xrightarrow{\varphi} H$

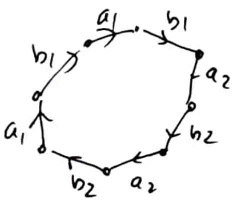
$\varphi([G,G]) \subset [\cancel{\varphi}H, H]$

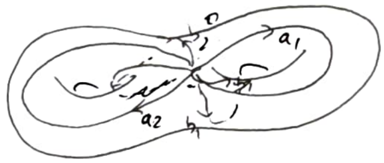
Other surfaces



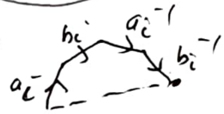


$KB = \mathbb{RP}^2 \# \mathbb{RP}^2$

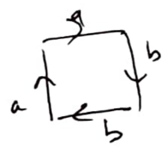




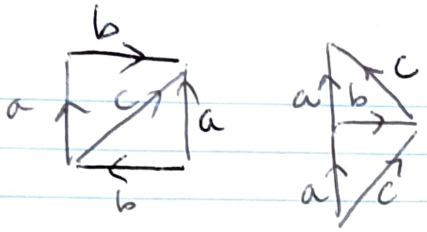
genus g g quadruples



M_g genus g $\pi_1(M_g) = \mathbb{Z}^{2g}$



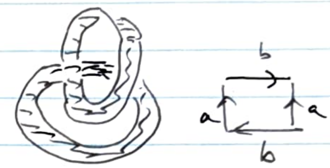
Follow pages 1–7 of Koch's notes "Classification of surfaces".



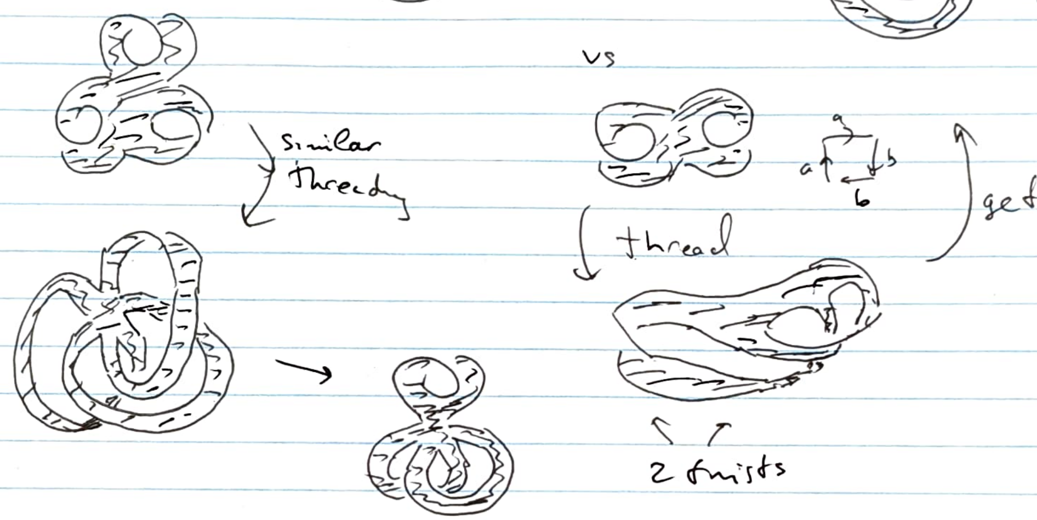
$KB \simeq \mathbb{RP}^2 \# \mathbb{RP}^2$

punctured T^2

punctured KB



for $3\mathbb{RP}^2$



$\Rightarrow \quad \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 \simeq T^2 \# \mathbb{RP}^2$

or see Koch's notes for a proof.