F - field, $f(x) \in F[x]$ irreducible $\Rightarrow$ $E = F[x]/(f(x))$ is a field

relabel x into α in definition of E $\qquad$ $E \supset F$ (extension of F)

and keep x as a formal variable.

$$\boxed{E = F[\alpha]/(f(\alpha))}$$

(Later:) E is a vector space over F with basis $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$, $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$, $\underline{\deg f = n}$

$\uparrow$ α direction, grow our field $\qquad$ all maps are inclusions $\hookrightarrow$

$$\begin{tikzcd} E = F[\alpha]/(f(\alpha)) \arrow[r, hook] & E[x] \\ F \arrow[u, hook] \arrow[r, hook] & F[x] \arrow[u, hook] \end{tikzcd}$$

$\boxed{\begin{array}{l} f(x) \text{ factors over } E,\ f(\alpha) = 0 \text{ in } E \Rightarrow \\ x - \alpha \mid f(x) \text{ in } E[x] \\ f(x) = (x-\alpha)g(x),\ g(x) \in E[x] \\ \deg g = n-1 \end{array}}$

$f(x)$ irreducible$/F$

$\boxed{\begin{array}{l} g(x) \text{ has } \underline{\text{complicated}} \text{ coefficients} \\ (\text{in } E, \underline{\text{not}} \text{ in } F) \end{array}}$

$\longrightarrow$ x direction, form polynomials

Examples:

1) $F = \mathbb{R}$, $f(x) = x^2+1$ $\qquad$ $E = \mathbb{R}[\alpha]/(\alpha^2+1) \simeq \mathbb{C}$ (relabel $\alpha \mapsto i$, $\mathbb{C} \simeq \mathbb{R}[i]/(i^2+1)$)
$i^2 = -1$

In $\mathbb{C}$ can factor $f(x) = x^2+1) = (x+i)(x-i)$

coefficients in $\mathbb{C}$, $\underline{\text{not}}$ in $\mathbb{R}$

2) $F = \mathbb{Q}$, $f = x^2 - 2$, $E = \mathbb{Q}[\alpha]/(\alpha^2-2)$ $\qquad$ since $E \subset \mathbb{R}$ (can realize E inside $\mathbb{R}$) can take $\alpha = \sqrt{2}$

$E \to \mathbb{R}$

$\alpha \mapsto \sqrt{2}$

$x^2 - 2 = (x-\alpha)(x+\alpha) = (x-\sqrt{2})(x+\sqrt{2})$

irr$/\mathbb{Q}$ $\qquad$ factors in E $\qquad$ factors in $\mathbb{R}$ $\qquad$ $F \subset E \subset \mathbb{R}$

3) $F = \mathbb{F}_2 = \mathbb{Z}/2$ $\quad$ $\{0, 1 \mid 1+1=0\}$ $\quad$ $f = x^2+x+1$ $\quad$ $E = \mathbb{F}_4 = \mathbb{F}_2[\alpha]/(\alpha^2+\alpha+1)$

$\boxed{+ = - \text{ in characteristic } 2}$ $\qquad$ $\alpha^{-1} = \alpha + 1 = \alpha^2$

$x^2+x+1 = (x+\alpha)(x+\alpha+1)$ $\qquad$ roots of f in $\mathbb{F}_4$: $\{\alpha, \alpha+1\}$

factors in $\mathbb{F}_4$

$0 \quad 1 \quad \alpha \to \alpha+1 = \alpha^2$

$\mathbb{F}_4^\times \simeq C_3$

4) $\underline{\text{Degree 3 example}}$ $\quad$ $F = \mathbb{Q}$, $f = x^3 - 2$ (irr$/\mathbb{Q}$, will see soon), $E = \mathbb{Q}[\alpha]/(\alpha^3-2)$ field, $x^3 - \alpha = (x-\alpha)(x^2+\alpha x+\alpha^2)$
$g(x)$, irred in E

$$\begin{array}{lcccc} \text{add root of } g(x) & K & = \mathbb{Q}[\sqrt[3]{2}, \omega\sqrt[3]{2}] & \subset & \mathbb{C} \\ & \cup & \cup & & \cup \\ \text{add root of } f(x) & E & \simeq \mathbb{Q}[\sqrt[3]{2}] & \subset & \mathbb{R} \\ & \cup & \cup & & \parallel \\ & F & = \mathbb{Q} & \subset & \mathbb{R} \end{array}$$

$\omega = e^{2\pi i/3}$ 3rd root of 1 $\qquad$ $K = E[\beta]/(\beta^2+\alpha\beta+\alpha^2)$

In K, $x^3 - 2 = (x-\alpha)(x-\beta)(x-\beta^2\alpha^{-1})$

$\beta \mapsto e^{2\pi i/3}\sqrt[3]{2}$

$\alpha \mapsto \sqrt[3]{2}$