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lecture 4

R - integral domain ($ab=0 \Rightarrow a=0$ or $b=0$; cancellation law holds)

(*) see page 2

$R \rightsquigarrow \mathrm{Frac}(R)$ or $Q(R)$ field of fractions.

$S = \{(a,b) \mid a,b \in R, b \neq 0\}$, equivalence relation

$(a,b) \sim (c,d) \Leftrightarrow ad = bc$.

$F = \mathrm{Frac}(R)$ is the set of equivalence classes

Define $+, \cdot$ on $\mathrm{Frac}(R)$, check that well-defined (do not depend on choices of representatives of equivalence classes.

$(a,b) \leftrightarrow \frac{a}{b}$ (easier to connect to usual fractions)

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Define $0 \in \mathrm{Frac}(R)$ as equivalence class $\{(0,b) \mid b \neq 0\}$.

Define $1 \in \mathrm{Frac}(R)$ as equivalence class $\{(a,a) \mid a \neq 0\}$.

<u>Theorem</u> 1) $\mathrm{Frac}(R)$ is a field with these binary operations $+, \cdot$ and $0, 1$ as above

2) $i: R \longrightarrow \mathrm{Frac}(R)$, $i(a) = (a,1)$ is an injective homomorphism of rings

<u>Proof</u>: part of 1) ~~was~~ is hw. exercise.

$F = \mathrm{Frac}(R)$ is abelian group under $+$, $\cdot$ is associative, distributivity, properties of 0 and 1.

$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$, exists iff $a \neq 0 \Leftrightarrow \frac{a}{b} \neq 0$.

Cancellation law in fractions $\frac{ar}{br} = \frac{a}{b}$ if $r \neq 0$

$\frac{a}{b} = \frac{c}{d} \iff ad = bc$

$\frac{a}{b} = \frac{ad}{bd} = \frac{bc}{bd} = \frac{c}{d}$

cancellation

write element $\frac{a}{b}$ as ab^{-1}, write $\frac{a}{1}$ as a.

$R \xrightarrow{i} \mathrm{Frac}(R)$. an injective homomorphism of rings

$i(a) = \frac{a}{1}$ $\quad$ i respects $+, \cdot$, takes 1 to 1 $\quad$ $i(1) = \frac{1}{1} = 1$

$i(a+b) = \frac{a+b}{1} = \frac{a}{1} + \frac{b}{1} = i(a) + i(b)$

$i(ab) = \frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1} = i(a)\, i(b)$

$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = a \cdot b^{-1}$

$\frac{b}{1} \cdot \frac{1}{b} = \frac{b}{b} = 1$ in $\mathrm{Frac}(R)$

Exercise: Prove injectivity.

Examples 1) $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$

2) If R is a field, $\mathrm{Frac}(R) = R$. Every nonzero element is already invertible.

Need an approach to easily prove statements like 2), see next page.

Prop (*) A finite integral domain is a field.

R an ID, $|R| < \infty$. Choose any $r \in R$, $r \neq 0$

$1, r, r^2, \ldots$ $\quad$ eventually $r^n = r^m$ for some $n < m$, $m = n + k$

$r^{n+k} = r^n$. Cancellation law $\Rightarrow r^k = 1 \Rightarrow r^{-1} = r^{k-1}$.

Characteristic of an integral domain $\nearrow p \iff R$ contains $\mathbb{Z}/p$

$\searrow \iff R$ contains $\mathbb{Z}$

injective homomorphism

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "i"'] & K \leftarrow \text{field} \\ \mathrm{Frac}(R) \arrow[ur, dashed] & \end{tikzcd}$$

$\exists!$ homomorphism $\tilde{\alpha}$ s.t. $\alpha = \tilde{\alpha}\, i$

<u>Prop</u> Given an injective homomorphism $\alpha: R \to K$ from an integral domain R to a field K, it extends uniquely to a homomorphism $\tilde{\alpha}: \mathrm{Frac}(R) \to K$, given by $\tilde{\alpha}\left(\frac{a}{b}\right) = \alpha(a)\alpha(b)^{-1}$.
$\tilde{\alpha}$ is injective.

<u>Proof</u> Define $\tilde{\alpha}\left(\frac{a}{b}\right)$ as $\alpha(a)\alpha(b)^{-1}$

(1) well-defined: if $\frac{a'}{b'} = \frac{a}{b}$ in $\mathrm{Frac}(R)$

$\tilde{\alpha}\left(\frac{a}{b}\right) = \tilde{\alpha}\left(\frac{a'}{b'}\right) \Leftrightarrow \alpha(a)\alpha(b)^{-1} = \alpha(a')\alpha(b')^{-1} \Leftrightarrow$
$\alpha(a)\alpha(b') = \alpha(a')\alpha(b) \Leftrightarrow \alpha(ab') = \alpha(a'b) \Leftrightarrow$
$ab' = a'b$ (need implications $\Leftarrow$)

(2) respects $+, \cdot$ $\qquad \tilde{\alpha}\left(\frac{a}{b} \cdot \frac{a'}{b'}\right) \overset{?}{=} \tilde{\alpha}\left(\frac{a}{b}\right)\tilde{\alpha}\left(\frac{a'}{b'}\right)$.

$\Downarrow$

<u>Exercise</u>: finish proving (2) $\qquad \tilde{\alpha}\left(\frac{aa'}{bb'}\right) = \alpha(a)\cdot\alpha(b)^{-1}\alpha(a')\alpha(b')^{-1}$

(3) takes $\frac{1}{1}$ to $1 \in K$ (clear). $\square$.

Get a model for $\mathrm{Frac}(R)$ given an injective homomorphism $\alpha: R \to K$ a field : take all el's $\alpha(a)\alpha(b)^{-1}$, $b \neq 0$.

If K ~~is~~ consists of these elements, $K \simeq \mathrm{Frac}(R)$.

Examples 1) $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$

2) $\mathrm{Frac}(\mathbb{Z}[\sqrt{2}]) = \mathbb{Q}[\sqrt{2}]$

3) $\mathrm{Frac}(\mathbb{Z}[i]) = \mathbb{Q}[i]$

4) $\mathrm{Frac}(\mathbb{Q}[x]) = \mathbb{Q}(x) = \left\{ \frac{f(x)}{g(x)} \,\middle|\, f(x), g(x) \in \mathbb{Q}[x],\ g(x) \neq 0 \right\} / \text{equiv.}$

Rings of functions

$R^X \leftarrow$ set $\qquad Y^X = \mathrm{Maps}(X, Y)$

n^m

$f(x)$ addition, multiplication - pointwise.

$\mathbb{R}^I \qquad I = [0,1)$.