Direct product of rings $R = R_1 \times R_2 = \{(a,b) \mid a \in R_1, b \in R_2\}$

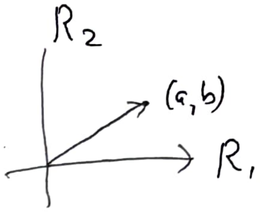
addition, multiplication - coordinatewise

$1 = (1,1)$ - unit element

$(a_1,b_1)+(a_2,b_2) = (a_1+a_2, b_1+b_2)$

$(1,0), (0,1)$ idempotents

$(a_1,b_1)(a_2,b_2) = (a_1a_2, b_1b_2)$



$1 = \underset{e_1}{\underset{\parallel}{(1,0)}} + \underset{e_2}{\underset{\parallel}{(0,1)}}$

e_1, e_2 - mutually orthogonal idempotents

$e_1e_2 = 0$, $1 = e_1+e_2$, $e_1^2 = e_1$, $e_2^2 = e_2$

$R_1 \xleftarrow{p_1} R_1 \times R_2 \xrightarrow{p_2} R_2$

$a \longleftarrow\!\shortmid (a,b) \longmapsto b$

p_1, p_2 - ring homomorphisms.

$R_1 \xrightarrow{i_1} R_1 \times R_2 \xleftarrow{i_2} R_2$

$a \longmapsto (a,0)$

$(0,b) \longleftarrow\!\shortmid b$

i_1, i_2 <u>non-unital</u> homomorphisms

$\underset{\text{unit el-t}}{1} \longmapsto \underset{e_1 \text{ idempotent}}{\underset{\parallel}{(1,0)}}$

$I \subset R_1 \times R_2$ ideal

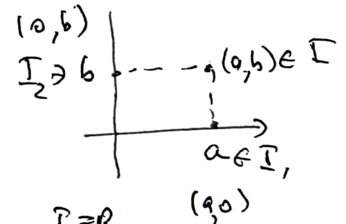
If $(a,b) \in I \Rightarrow$

$(1,0)(a,b) = (a,0) \in I$

$(0,1)(a,b) = (0,b) \in I$.

$\Rightarrow$ I has a product structure, $I = I_1 \times I_2$, $I_1 \subset R_1$, $I_2 \subset R_2$

$I_1 \subset R_1$ an ideal, $I_2 \subset R_2$ an ideal



This is a description of ideals in $R_1 \times R_2$.

$R_1 \times R_2 / I \simeq R_1/I_1 \times R_2/I_2$

$\nwarrow$ not an ID, unless $R_1/I_1 = 0$ or $R_2/I_2 = 0$

$I_1 = R_1 \Updownarrow$ $I_2 = R_2 \Updownarrow$

If $I_2 = R_2$, $R/I = R_1/I_1$ ID iff $I_1 \subset R_1$ prime, field iff $I_1 \subset R_1$ maximal, likewise if $I_1 = R_1$

$\Rightarrow$ Description of prime ideals in $R_1 \times R_2$:

A prime ideal $I = I_1 \times R_2 = I_1 \times (1)$, where $I_1 \subset R_1$ is prime or

$I = R_1 \times I_2 = (1) \times I_2$, where $I_2 \subset R_2$ is prime.

Likewise for max. ideals.