Modern Algebra II, spring 2022

Homework 9, due Wednesday April 6.

1.(20 points) Which of the following numbers are constructible using a ruler and compass? Briefly justify your answer.

$$\frac{1}{2}\sqrt[3]{3}, \qquad \sqrt{6+\sqrt{7}}, \qquad \sqrt[4]{5}-1, \qquad \sqrt[6]{2}+1.$$

2.(20 points) Briefly sketch the steps involved into constructing numbers $\sqrt{2}$, $\sqrt{\sqrt{2}+1}$ and $\sqrt{\sqrt{\sqrt{2}+\sqrt{3}}+1}$ using a ruler and compass.

3.(20 points) Suppose we have a ruler and compass, as before, but are given 3 points A, B, C on a line in the plane, with B between A and C and distances $|AB| = 1$, $|BC| = \sqrt[3]{2}$. Explain how to modify the arguments in this week's lectures to show that $\sqrt[5]{2}$ is not constructible with these assumptions. (Hint: What are the properties of the tower of fields $\mathbb{Q} \subset K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n$ where the field K_i is generated by the coordinates of A, B, C and of the next i points that we create? What can you say about the degree $[K_n : \mathbb{Q}]$?)

4.(20 points) (a) Recall the definition of a normal extension E/F (or see our usual references). Explain in your own words what is an obstacle for an extension to be normal.

(b) Let E/F be a degree two extension. Prove that E is normal. Hint: pick an element $\alpha \in E \setminus F$. Write down its irreducible polynomial $f(x)$. Can you show that E is a splitting field of $f(x)$? You need to check that E contains all roots of $f(x)$, not just α.

(c) Look through class notes and find an example of degree 3 extension of $\mathbb{Q}$ which is not normal. Generalize that example and describe a degree n extension of $\mathbb{Q}$ which is not normal, for any $n \geq 3$.

(d) Explain why any extension of finite fields $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ is normal.

5.(20 points) For any automorphism σ of a ring R we can define the subring R^σ of elements fixed by σ.

(a) Give a definition of R^σ using mathematical notations (via sets and quantifiers) and prove that R^σ is a subring.

(b) Suppose $R = F[x]$, where F is a field, and σ takes a polynomial $f(x)$ to $f(-x)$. For instance, if $f(x) = a + bx + cx^2$, then $\sigma(f)$ is the polynomial $a - bx + cx^2$. Prove that the subring R^σ of polynomials invariant under σ (equivalently, fixed by σ) is the subring $F[x^2]$ if $\operatorname{char} F \neq 2$. What happens when F has characteristic two?