Modern Algebra II, Spring 2022

Homework 3, due Wednesday February 9.

This week we've covered integral domains, their fields of fractions, homomorphisms and ideals, and quotient rings. (We stated by did not prove the First Isomorphism Theorem for rings). You can find these topics in Rotman (pages 13-23), in Howie (Section 1.2-1.4), and in R.Friedman's notes Integral Domains (see webpage).

As before, all rings are assumed commutative in this homework.

1. (15 points) Which of the following are ideals? (Brief explanations are welcome but not required.)
(a) Subset $2\mathbb{Z}$ of all even integers in $\mathbb{Z}$.
(b) Polynomials in $R[x]$ with zero constant term ($a_0 = 0$).
(c) Subset $R \times 0$ of the direct product $R \times S$ of two rings R, S.
(d) The set of diagonal elements $\Delta = \{(x, x) \ : \ x \in R\}$ in the direct product $R \times R$ of a ring R with itself.
(e) The set R^* of invertible elements of a ring R.
(f) Polynomials in $\mathbb{Z}[x]$ with all coefficients even.

2. (10 points) (a) Let N be the set of nilpotent elements in a ring R. Prove that N is an ideal in R. (Problem 6c from Homework 2 is a part of this statement.)
(b) Describe the ideal N when $R = \mathbb{Z}/8$ is the ring of residues modulo 8.

3. (15 points) (a) Prove that the sum $I + J$ and intersection $I \cap J$ of ideals of R is an ideal of R (we briefly discussed this in class).
(b) Given elements $a_1, \ldots, a_n \in R$, check that the set

$$Ra_1 + \cdots + Ra_n \ := \{r_1 a_1 + \cdots + r_n a_n | r_1, \ldots, r_n \in R\}$$

in an ideal of R. (We call it the ideal generated by $a_1, \ldots, a_n$ and denote $(a_1, \ldots, a_n)$.)

4. (10 points) Which of the following rings are integral domains?

$$\mathbb{R},\ \mathbb{Z}/20,\ \mathbb{C}[x],\ \mathbb{Z}/19,\ \mathbb{R} \times \mathbb{Q},\ \mathbb{Q}[x, y]$$

5. (10 points) Compute the following ideals of $\mathbb{Z}$. (Each of them is a principal ideal of the form (n), for some $n \geq 0$. Find n in each case.) Which of these ideals are proper?

$$(4) + (3),\ (5) + (0),\ (10) + (5),\ (6) \cap (3),\ (10) \cap (4),\ (3) \cap (0).$$

6. (10 points) Using the results proved in class, explain why
(a) $\mathbb{Q}$ is the field of fractions of $\mathbb{Z}[\frac{1}{n}]$,

(b) $\mathbb{Q}[\sqrt{3}]$ is the field of fractions of $\mathbb{Z}[\sqrt{3}]$.

(Hint: in each case show that there is an injective homomorphism (inclusion) α from the first ring to the second, that the second ring is a field which is generated by elements of the form $\alpha(a)$ and $\alpha(b)^{-1}$ for a, b in the first ring.

7. (*Optional problem*, 10 points) Let $e \in R$ be an idempotent.

(a) Check that Re and $R(1-e)$ are ideals of R and that their intersection $Re \cap R(1-e) = 0$. Show that any element $a \in R$ has a unique presentation as a sum of an element in Re and an element in $R(1-e)$.

(b) Prove that Re is a ring, with identity e and addition and multiplication inherited from R. Likewise for $R(1-e)$. (Since $1-e$ is also an idempotent, you don't need to repeat your arguments twice.)

(c) Show that the map from $Re \times R(1-e)$ to R that takes (a, b) to $a+b$ is an isomorphism of rings.

This exercise tells you that an idempotent in a (commutative) ring allows you to decompose the ring as a direct product.

Additional problem for practice, won't be graded.

I. Show that the ideal $(2, x)$ in $\mathbb{Z}[x]$ is not principal. (Hint: assume it is principal, generated by a polynomial $f(x)$. What can you say about the degree of f?)