Topics in Stochastic Geometry

Convex Hulls of Random Walks

Lecture Notes for GR8260, Spring 2020, Columbia University

Julien Randon-Furling

Classes: Tue & Thu 2:40-3:55pm, Mathematics Hall Room 622
Office hours (Room 625): Mon to Fri 9am - 2pm or appointment

Abstract This course covers a range of results on the convex hull of random walks in the plane and in higher dimension: expected perimeter length in the planar case, expected number of faces on the boundary, expected \(d\)-dimensional volume, and other geometric properties of such random convex polytopes.

Beforehand, we will review a number of classical one-dimensional results such as the Arcsine laws. Moving on to higher-dimensional random walks, we will introduce the fundamental object of this course, that is the joint convex hull of a collection of \(m\) random walks:

\[
C_d = \operatorname{conv} \left( 0, S_1^{(1)}, \ldots, S_{n_1}^{(1)}, \ldots, S_1^{(m)}, \ldots, S_{n_m}^{(m)} \right),
\]

with \((S_1^{(l)})_{i=1}^{n_1}, \ldots, (S_1^{(m)})_{i=1}^{n_m}\) defined as

\[
S_i^{(l)} = X_1^{(l)} + \cdots + X_i^{(l)}, \quad 1 \leq l \leq m, \quad 1 \leq i \leq n_l,
\]

where \(m, n_1, \ldots, n_m \in \mathbb{N}\), and

\[
X_1^{(1)}, \ldots, X_{n_1}^{(1)}, \ldots, X_1^{(m)}, \ldots, X_{n_m}^{(m)}
\]

are independent \(d\)-dimensional random vectors drawn from the same distribution (eg Gaussian).

The case \(m = 1\) is the \(d\)-dimensional random walk; the case \(m \geq d + 1\) and \(\forall l \geq 1, n_l = 1\) is, when the underlying distribution is the Gaussian one, the standard Gaussian polytope. Other cases include multiple, independent \(d\)-dimensional random walks.

In dimension 2 the course will cover results such as the Spitzer-Widom formula for the expected circumference of the convex hull, and Baxter’s combinatorial lemma for the expected number of edges on the boundary of the convex hull. In higher dimension, we will examine recent results on the expected number of faces of the boundary of the convex hull, including the case of multiple random walks.

In the last part of the course, we will discuss related topics such as the convex hull of Brownian
motion (in the plane and in higher dimensions) together with connections to one-dimensional results on the greatest convex minorant of random walks, Brownian motion and more general Lévy processes. While doing so, we will review classical results on excursions, meanders and bridges.

**Prerequisite**
Basic knowledge of probability theory, random walks, stochastic processes.

**Suggested reading**

1 Positive points on the path of a random walk

We start by reviewing a fundamental theorem pertaining to the number of positive points on the path of a one-dimensional random walk. Namely, if \( X_1, X_2, \ldots \) are real random variables (at this stage we do not make any assumption on their joint distribution), one defines the corresponding random walk as

\[
S_n = X_1 + X_2 + \cdots + X_n,
\]

and we set \( S_0 = 0 \).

It is customary to call \( X_1, X_2, \ldots \) the “jumps” of the random walk, and \( S_1, S_2, S_3, \ldots, S_n \) the “positions” or simply the “points” of the random walk. The set \( \{(k, S_k), 0 \leq k \leq n\} \) is often referred to as the “path” of the random walk, that is, its set of “steps”. If we prefer viewing a random walk simply as a sum of random variables, then \( S_0, S_1, S_2, S_3, \ldots, S_n \) is just the sequence of partial sums.

Now define the following random variable:

\[
N_n = \sum_{k=0}^{n} 1\{S_k > 0\},
\]

i.e. the number of positive points in the first \( n \) steps of the random walk (\( 1 \) denotes an indicator function). The law of \( N_n \) was computed by Erik Sparre Andersen in 1949. Before we state his result, we need the following definition.

**Definition 1.** An \( s \)-permutation \((x_1, \ldots, x_n)\) of \( n \) real numbers \((a_1, \ldots, a_n)\) is a permutation \((\varepsilon_i a_i, \ldots, \varepsilon_n a_n)\) of the \( n \) numbers \((\varepsilon a_1, \ldots, \varepsilon a_n)\) where for all \( 1 \leq i \leq n \), \( \varepsilon_i = \pm 1 \).

Note that there are \( 2^n n! \) \( s \)-permutations.

**Theorem 1.** Let \( X_1, X_2, \ldots \) be real random variables with a joint distribution admitting a density \( f_n : \mathbb{R}^n \to \mathbb{R}_+ \) invariant under \( s \)-permutations. Then, for \( 0 \leq m \leq N \),

\[
P[N_n = m] = (-1)^n \left( \frac{-\frac{1}{2}}{m} \right) \left( \frac{-\frac{1}{2}}{n-m} \right),
\]

and \( P[N_n = m] = 0 \) for other values of \( m \).

Here \( \binom{z}{k} \) is the generalized binomial coefficient, i.e. equal to \( z(z-1) \cdots (z-k+1) / k! \).

Before we prove Theorem 1, let us try and understand why an invariance under \( s \)-permutations is asked for, and how the proof will rely on it. First, trivially:

\[
P[N_n = m] = \mathbb{E} [g_{n,m}(X_1, \ldots, X_n)] = \int_{\mathbb{R}^n} g_{n,m}(x_1, \ldots, x_n) f_n(x_1, \ldots, x_n) \, dx_1 \ldots dx_n
\]

where \( g_{n,m} : \mathbb{R}^n \to \{0,1\} \) is the indicator function of the set \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_n) \text{ has } m \text{ partial sums}\} \).

Now, let us notice the following fact: given a point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), the corresponding partial sums — i.e. \( s_0 = 0, s_1 = x_1, s_2 = x_1 + x_2, \text{etc.} \) — may be written as the standard dot products between \( x \) and the vectors \( \eta_1, \eta_2, \ldots, \eta_n \), defined as \( n \)-tuples with 1s in the first \( k \) components and only 0s afterwards. Therefore, the \( k \)-th partial sum of \( x \) will be positive if and only if, in the Euclidean space \( \mathbb{R}^n \), \( x \) lies on the same side as \( \eta_k \) of the hyperplane \( H_{\eta_k} \) orthogonal to \( \eta_k \). This is a simple geometric criterion for counting positive partial sums.
From the geometric criterion, we see that the law of $N_n$ will be given by the integral of $f_n$ on regions delineated by the finite family of hyperplanes $H_{\eta_k}$, because on such domains $g_{n,m}$ is identically 0 except for one value of $m$. Hence, any choice of $f_n$ that leaves the probability contents of these regions invariant will lead to the same law for $N_n$. Unfortunately, when $n > 2$, it is not the case that each of these regions may be built from a fundamental partition of the space $\mathbb{R}^n$ into cells or chambers obtained from the full family of “coordinate” and “diagonal” hyperplanes: $H_\eta, \eta = \pm 1$ and $1 \leq \sum_{i=1}^{n} \eta_i^2 \leq 2$. These hyperplanes generate the Weyl group of the root system of type $B_n$, and the disjoint cells separated by the hyperplanes are called Weyl chambers. There are $2^n n!$ such chambers. When $f_n$ is invariant under s-permutations, the integral of $f_n$ on any two of the Weyl chambers is the same – thus, each Weyl chamber carries the same probability, viz. $1/[2^n n!]$.

Though $g_{n,m}$ is not constant on each Weyl chambers, it can be proved (and we did it in class) that if we let

$$G_{n,m}(a_1, \ldots, a_n) = \sum_{\varepsilon_i = \pm 1, \sigma \in S_n} g_{n,m}(\varepsilon_{\sigma(1)}a_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)}a_{\sigma(n)}),$$

then $G_{n,m}$ is constant in $\mathbb{R}^n \setminus \bigcup_{\eta \in \{-1,0,1\}^n} H_\eta$. The constant value of $G_{n,m}$ may be obtained by looking at whichever point $(a_1, \ldots, a_n)$ we like, and counting how many of its $2^n n!$ images (including itself) under s-permutations have exactly $m$ positive partial sums.

**Lemma 1.** Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be such that $\forall \eta \in \{-1,0,1\}^n$, $\eta_1a_1 + \cdots + \eta_n a_n \neq 0$, except for $\eta = 0$. Then the number of s-permutations of $(a_1, \ldots, a_n)$ for which exactly $m$ partial sums are positive depends only on $n$ and $m$, and is equal to

$$p_{n,m} = \frac{n!}{2^m} \binom{2m}{m} \binom{2(n-m)}{n-m} = 2^n n! (-1)^n \left(\frac{-1}{2}\right)^m \left(\frac{-1}{2}\right)^{n-m}.$$

**Proof.** Let us pick a particular choice of $a$, one that satisfies

$$a_1 > 0, a_2 > a_1, a_3 > a_1 + a_2, \ldots, a_n > a_1 + \cdots + a_{n-1}.$$  

Thus the sign of a partial sum of an s-permutation of $a$ is easily determined by just looking at the sign of the term with highest index. Now, observe that one may obtain an s-permutation $(x_1, \ldots, x_n)$ of $(a_1, \ldots, a_n)$ with $m$ positive partial sums as follows. Pick $m$ numbers $a_{j_1}, \ldots, a_{j_m}$ that will be the last terms (with plus or minus sign) in $m$ positive partial sums. The remaining numbers $a_{j_{m+1}}, \ldots, a_{j_n}$ will be last terms in negative partial sums. There are $\binom{n}{m}$ possible choices at this stage.

If $a_n$ is amongst $a_{j_1}, \ldots, a_{j_m}$ then it must appear with positive sign in $(x_1, \ldots, x_n)$ since $a_n > a_1 + \cdots + a_{n-1}$. Then $x_1 + \cdots + x_n > 0$, and therefore $|x_n|$ is also one of the $a_{j_1}, \ldots, a_{j_m}$ — and so possibly equal to $a_n$. If it is the case, then $x_n$ cannot be negative; but if $|x_n| \neq a_n$, $x_n$ might be positive or negative. That is, $2m - 1$ possibilities here.

If $a_n$ is not amongst $a_{j_1}, \ldots, a_{j_m}$, then $|x_n|$ is one of $a_{j_{m+1}}, \ldots, a_{j_n}$ and there are similarly $2(n-m) - 1$ possibilities for $x_n$.

And so on for $x_{n-1}, x_{n-2}, \ldots, x_1$.

Eventually, to fix a given s-permutation $(x_1, \ldots, x_n)$, one has chosen $m$ times from $a_{j_1}, \ldots, a_{j_m}$ and $n - m$ times from $a_{j_{m+1}}, \ldots, a_{j_n}$. This leads to

$$p_{n,m} = \binom{n}{m} (2m - 1) (2m - 3) \cdots \times 3 \times 1 \times (2(n-m) - 1) (2(n-m) - 3) \cdots \times 3 \times 1$$

$$= 2^n n! (-1)^n \left(\frac{-1}{2}\right)^m \left(\frac{-1}{2}\right)^{n-m}.$$  

1Note that the integral of $f_n$ on the union of the hyperplanes themselves will be 0 since we are integrating with respect to the Lebesgue measure in $\mathbb{R}^n$.  

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Using the Chu-Vandermonde identity, one checks easily that \( \sum_{m=0}^{n} p_{n,m} = 2^n n! \).

From Lemma 1 and the discussion immediately above it, it follows that

\[
P[N_n = m] = (-1)^n \left( \frac{-1}{2} \right)^m \left( \frac{-1}{2} \right)^{n-m},
\]

as stated in Theorem 1.

**Exercise:** Show that, under the assumptions of Theorem 1,

\[
\lim_{n \to \infty} P \left( \frac{N_n}{n} \leq \alpha \right) = \frac{2}{\pi} \arcsin \sqrt{\alpha},
\]

for any \( \alpha \in [0, 1] \).

This generalizes a result established in 1947 by P. Erdős and M. Kac for random walks with independent, identically distributed jumps having expectation 0 and variance 1 (*Bull. Am. Math. Soc.* **53**, 1011-1020). As one might guess, the Arcsine law is also that of the sojourn time of a Brownian motion on the positive side of its origin (Lévy, 1939).
2 Number of positive points & index of the maximum

As in the previous section, let $X_1, X_2, \ldots$ be real random variables with a joint distribution admitting a density $f_n : \mathbb{R}^n \to \mathbb{R}_+$ invariant under s-permutations. Let $N_n$ be the number of positive points on the path of the random walk.

From Theorem 1, observe that

$$P[N_n = m] = P[N_m = m] P[N_{n-m} = 0].$$

In words: the probability to have $m$ positive points on the path is equal to the probability obtained when combining two independent subwalks, one with $m$ steps and only positive points and one with $n-m$ steps and only negative points! This is all the more surprising as this should not be mistaken with the probability that the first $m$ points of the walk be positive and the remaining $n - m$ ones be negative — the number of positive points after the $k$-th step will indeed in general depend on the position $S_k$. The observation would in fact make sense more easily if instead of $N_n$ we were considering a random variable for which there is some form of independence between the subwalk up to an arbitrary step $k$ and the subwalk after $k$. Think for instance of the index of the maximal position: if you split the path at some step $k$, the index of the maximal position in the subwalk up to step $k$ is independent of the index of the maximal position in the subwalk after step $k$. The observation from Theorem 1 thus hints at a surprising and beautiful result, also established by Andersen. We state and prove it here for independent, identically distributed jumps, but it is also valid for non-separable joint jump distribution invariant under s-permutations. It is not true for non separable symmetric joint jump distributions (i.e. invariant solely under permutations but not s-permutations).

**Theorem 2. Principle of Equivalence** Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed real random variables. Let $N_n$ be the number of positive points on the corresponding random walk, and $K_n$ be the index of the step when the maximal position is reached (thus $S_i < S_{K_n}$ for all $i < K_n$, and $S_i \leq S_{K_n}$ for all $i > K_n$). Then, for any integer $m$,

$$P[N_n = m] = P[K_n = m].$$

**Proof.** In the same spirit as the proof of Theorem 1, we give a purely combinatorial proof. (This proof appears in W. Feller’s book, and was discovered by A.W. Joseph — who seems to have not published it himself, although he did publish another proof of the same result, for which he credits M.T.L. Bizley.) Given $(a_1, \ldots, a_n)$ real numbers, let $A_r$ be the number of permutations of these numbers for which exactly $r$ partial sums are positive (with $0 \leq r \leq n$), and let $B_r$ be the number of permutations for which the maximal value of the partial sums is (first) realized by the $r$th partial sum.

We prove by induction on $n$ that $A_r = B_r$ for all $0 \leq r \leq n$. Clearly, when $n = 1$, $A_0 = B_0$ and $A_1 = B_1$. Suppose the property holds for $n - 1 \geq 1$. Let $A_r^{(k)}$ and $B_r^{(k)}$ be defined as $A_r$ and $B_r$ but solely on the $(n-1)$-tuple obtained when removing $a_k$ from the list. By the induction hypothesis, $A_r^{(k)} = B_r^{(k)}$ for all $0 \leq r \leq n - 1$. Also, for $r = n$, $A_n^{(k)} = B_n^{(k)} = 0$. Now:

(i) if $a_1 + \cdots + a_n \leq 0$, then $A_n = B_n = 0$, and for any permutation the number of positive partial sums and the (first) index of the maximum will only depend on the first $n - 1$ elements. Since the $n!$ permutations of $(a_1, \ldots, a_n)$ may be partitioned according to which element is put in the last place, we therefore find that, for any $0 \leq r \leq n - 1$,

$$A_r = \sum_{k=1}^{n} A_r^{(k)} \text{ and } B_r = \sum_{k=1}^{n} B_r^{(k)},$$

which leads to $A_r = B_r$ by the induction hypothesis.

(ii) if $a_1 + \cdots + a_n > 0$, then for any $1 \leq r \leq n$ one has that $A_r = \sum_{k=1}^{n} A_r^{(k)}$. For $B_r$, let us
partition the set of all permutations of \((a_1, \ldots, a_n)\) according to the element that is put in the first place. Then, the subwalk constructed with the \(n - 1\) next elements reaches its maximum at its \(r\)-th step if and only if the full random walk (with the first element) reaches its maximum at step \(r + 1\). Indeed, when the first element is positive, this will just shift the subwalk upward, preserving the position of its maximum; and if the first element is non-positive, then it cannot be “too” negative: if it were less than the opposite of the maximum of the subwalk, then the final position could not be positive, in contradiction with \(a_1 + \cdots + a_n > 0\). Therefore, one also has that \(B_r = \sum_{k=1}^{n} B^{(k)}_{r-1}\). Hence, by the induction hypothesis, again \(A_r = B_r\) for all \(r\).

We leave the rest of the proof as an exercise for the reader.

Hence, to prove directly the observation made from Theorem 1, it now suffices to establish it for \(K_n\).

**Theorem 3.** Let \(X_1, X_2, \ldots, X_n\) be independent, identically distributed real random variables. Let \(K_n\) be the index of the step when the maximal position is reached (thus \(S_i < S_{K_n}\) for all \(i < K_n\), and \(S_i \leq S_{K_n}\) for all \(i > K_n\)). Then, for any integer \(m\),

\[
P[K_n = m] = P[K_m = m] P[K_{n-m} = 0].
\]

**Proof.** Note that by definition of \(K_n\),

\[
P[K_n = m] = P \left[ \bigcap_{i=0}^{m-1} \{S_i < S_m\} \cap \bigcap_{i=m+1}^{n} \{S_i \leq S_m\} \right].
\]

Hence

\[
P[K_n = m] = P \left[ \{K_m = m\} \cap \bigcap_{i=m+1}^{n} \{X_{m+1} + \cdots + X_{m+i} \leq 0\} \right].
\]

\[
= P \{{K_m = m}\} \times P \left[ \bigcap_{i=m+1}^{n} \{X_{m+1} + \cdots + X_{m+i} \leq 0\} \right]
\]

\[
P[K_n = m] = P[K_m = m] P[K_{n-m} = 0],
\]

where we have used the independence of \(X_1, \ldots, X_n\).
3 Edges on the convex hull of a planar random walk

We now move on to planar random walks, with a result that also relies on making use of permutations thanks to multiple symmetries.

Let us consider vectors in the plane: $z_1, \ldots, z_n \in \mathbb{R}^2$. Let $z = z_1 + \ldots + z_n$, and for $1 \leq k \leq n$, we also write as before the partial sums $s_k = z_1 + \cdots + z_k$. We will not need here $s$-permutations but only standard permutations, and for $\sigma \in S_n$ we write

$$s_k(\sigma) = z_{\sigma(1)} + \cdots + z_{\sigma(k)}.$$ 

For $A \subset 1, \ldots, n$, we let $z_A = \sum_{i \in A} z_i$. We also introduce a notion that plays the same role here as the condition in Theorem 1 in the first section.

**Definition 2.** The family of vectors $z_1, \ldots, z_n$ is said to be skew if:

$$\forall A, B \subset \{1, \ldots, n\}, z_A = z_B \Rightarrow A = B.$$ 

Under this condition, for a given choice of $A \subset 1, \ldots, n$, the number of permutations for which $z_A$ appears as an edge on the boundary of the convex hull of the path $\{0, s_{\sigma(1)}, \ldots, s_{\sigma(n)}\}$ depends only on $n$ and the cardinal of $A$, as we show now.

**Lemma 2.** Let $z_1, \ldots, z_n$ be skew. Then

1. There exists a unique cyclic permutation $\rho \in S_n$ such that the partial sums $s_{\rho(1)}, \ldots, s_{\rho(n)}$ all lie on the same side of the line with direction $z$.

2. Let $A \subset 1, \ldots, n$ with $m$ elements. Then $z_A$ appears as an edge on the boundary of the convex hull for exactly

$$2(m-1)! (n-m)!$$

of the $n!$ paths $\{0, s_{\sigma(1)}, \ldots, s_{\sigma(n)}\}$ obtained as $\sigma$ ranges over $S_n$.

**Proof.**

1. $z_1, \ldots, z_n$ skew $\Rightarrow$ there is at most one point among $\{0, s_1, \ldots, s_{n-1}\}$ which lies at a maximum distance from the line with direction $z$. Say this point is $s_k$. Then take:

$$\rho(1) = k + 1, \rho(2) = k + 2, \ldots, \rho(n-k) = n, \rho(n-k+1) = 1, \ldots, \rho(n) = k.$$ 

This is a cyclic permutation of $z_1, \ldots, z_n$, with all partial sums on the same side of the line with direction $z$. The uniqueness of $\sigma$ thus defined follows from that of $k$ – and therefore from the skewness of $z_1, \ldots, z_n$.

Note: if one quotients $S_n$ by the equivalence relation corresponding to cyclic permutations (i.e. two permutations $\sigma_1, \sigma_2$ are in the same equivalence class if and only if there exists a cyclic permutation $\rho$ such that $\sigma_1 = \rho \circ \sigma_2$), the result above means that there is only one permutation per class for which all points lie on the same side of the line given by $z$. Hence there are in total $(n-1)!$ permutations $\sigma$ such that $s_{\sigma(1)}, \ldots, s_{\sigma(n)}$ lie on the same side of the line (since there are $n$ permutations in each class, and of course $n!$ elements in $S_n$).

2. Let $z_{n+1} = -s_n = -z$ and, given $A \subset 1, \ldots, n$ with $m$ elements, let

$$A' = \{1, \ldots, n, n+1\} \setminus A.$$ 

Also, $s_{n+1}(\sigma) = s_0(\sigma) = 0$ for all $\sigma \in S_n$.

Note that $z_A$ appears as an edge on the boundary of the convex hull of $\{s_{\sigma(0)}, s_{\sigma(1)}, \ldots, s_{\sigma(n)}\}$

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if and only if there exists 0 \leq k \leq n - m such that \( z_A = s_{k+m}(\sigma) - s_k(\sigma) \) and all partial sums of \( \{ z_{\sigma(k+1)}, \ldots, z_{\sigma(k+m)} \} \) lie on the same side of the line given by \( z_A \) and all partial sums of \( \{ z_{\sigma(n)}, z_{n+1}, z_{\sigma(1)}, \ldots, z_{\sigma(k)} \} \) lie also on that same side of the line.

From the first point, we have that there will be \( 2 \times (m - 1)! \times (m - n)! \) permutations for which such a configuration happens. (The factor 2 comes from the fact that there are two sides to a line.)

To apply the preceding lemma to a planar random walk, note that if \( Z_1, \ldots, Z_n \) are independent, identically distributed random vectors all with a density with respect to the Lebesgue measure, then almost surely they form a skew family.

**Definition 3.** For \( Z_1, \ldots, Z_n \) independent, identically distributed random vectors, define

\[
C_n = \text{conv} \left( 0, S_1, \ldots, S_n \right),
\]

the convex hull of the corresponding random walk, and \( \partial C_n \) its boundary. Note that almost surely \( \partial C_n \) is a finite union of edges (straight line segments).

Let \( K_n = \text{Card} \left( \{ i \mid Z_i \in \partial C_n \} \right) \), where by \( Z_i \in \partial C_n \) we mean that the vector \( Z_i \) is an edge on the boundary of the convex hull; \( V_n = \text{Card} \left( \{ \text{edges of } \partial C_n \} \right) = \text{Card} \left( \{ \text{vertices on } \partial C_n \} \right) \); and \( L_n = \text{Length of the perimeter of } C_n = \text{Length of } \partial C_n \).

Letting \( W_n \) be any of \( K_n, V_n \) or \( L_n \) and writing \( W_n(\sigma) \) for \( W_n \) computed on the random walk constructed from \( Z_{\sigma(1)}, \ldots, Z_{\sigma(n)} \), one finds that

\[
\mathbb{E} \left[ \sum_{\sigma \in S_n} W_n(\sigma) \right] = n! \mathbb{E}[W_n],
\]

as \( Z_1, \ldots, Z_n \) are independent, identically distributed random vectors.

Now we are ready to prove the following theorem.

**Theorem 4.**

\[
\mathbb{E}[K_n] = 2,
\]

\[
\mathbb{E}[V_n] = 2 \sum_{k=1}^{n} \frac{1}{k},
\]

\[
\mathbb{E}[L_n] = 2 \sum_{k=1}^{n} \frac{\mathbb{E}[\|S_k\|]}{k}.
\]

**Proof. Exercise**

The previous theorem is in fact a special case of the next one, with a similar proof.

**Theorem 5.** Writing \( l_1, \ldots, l_{V_n} \) for the lengths of the edges on \( \partial C_n \) (numbered eg counterclockwise from the one intersecting a preferred half-line from the origin), we define for any real valued function \( f \) the random variable \( F_n = \sum_{k=1}^{V_n} f(l_k) \). Then, one has that

\[
\mathbb{E}[F_n] = 2 \sum_{k=1}^{n} \frac{\mathbb{E}[f(\|S_k\|)]}{k}.
\]

2In fact, we see here that it is sufficient to have \( Z_1, \ldots, Z_n \) exchangeable random vectors, each with at least one of their components admitting a density (so that they form a.s. a skew family).
4 Some combinatorial lemmas in higher dimensions

One way to extend the results in the previous section to higher dimensional random walks is to establish combinatorial lemmas similar to the one on which these results rely.

**Lemma 3.** Let $z_1, \ldots, z_n$ be a family of $p$-dimensional vectors ($n \geq p$). Let $H$ be a hyperplane which contains $z = z_1 + \cdots + z_n$ but no other sum of any subset of the family. Let also $C$ denote one of the half-spaces determined by $H$. Then, there exists exactly one cyclic permutation $\rho$ such that exactly $m$ of the partial sums $s_1(\rho), \ldots, s_{n-1}(\rho)$ lie in $C$.

**Proof.** Let $u$ be a unit vector normal to $H$. There exists exactly one index $k$ such that the scalar product $n.s_k$ is minimal. Then simply set $\rho(1) = k + 1, \ldots, \rho(n - k) = n, \rho(n - k + 1) = 1, \ldots, \rho(n) = k$.

And the uniqueness of $\rho$ follows from that of $k$. \hfill \square

Before the next lemma, we extend further the one-dimensional notion of skewness for a family of vectors.

**Definition 4.** We say that $z_1, \ldots, z_n$ forms a skew family if and only for any non-empty disjoint subsets $A_1, A_2, \ldots, A_p$ of $1, \ldots, n$ the vectors $z_{A_1}, \ldots, z_{A_n}$ are linearly independent. (Recall that $z_{A_j} = \sum_{i \in A_j} z_i$.)

**Lemma 4.** Let $A_1, A_2, \ldots, A_{p-1}$ be non-empty disjoint subsets of $1, \ldots, n$. Then the vectors $z_{A_1}, \ldots, z_{A_{p-1}}$ appear on the boundary of the convex hull of exactly

$$2 (n_{A_1} - 1)! (n_{A_2} - 1)! \cdots (n_{A_{p-1}} - 1)! (n - n_{A_1} - \cdots - n_{A_{p-1}})!$$

paths out of the $n!$ paths $s_0(\sigma), s_1(\sigma), \ldots, s_n(\sigma)$ as $\sigma$ ranges over all permutations.

**Proof.** Exercise. \hfill \square