Background Material for Morse Theory Lectures

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1 Isotopy

Given a homotopy $H\colon X \times [0,1] \to Y$, for every $t \in [0,1]$ we denote by $H_t\colon X \to Y$ the restriction of H to $X \times \{t\}$.

Let $f\colon X \to M$ be a smooth embedding of one manifold in another. An *isotopy* of f is a smooth map. $F\colon X \times I \to M$ with $F_0 = f$ and with F_t a smooth embedding for all $t \in [0,1]$.

We say that f and g are *isotopic* if there is an isotopy $F\colon X \times I \to M$ with $F_0 = f$ and $F_1 = g$, and we say that F *is an isotopy from* f *to* g.

Claim 1.1. *Isotopy is an equivalence relation on the space of smooth embeddings of* X *into* M.

Proof. The only thing that is not immediate is that the relation is transitive. Suppose that F is an isotopy from f to g and G is an isotopy from g to h. Fix a C^∞-map $\mu\colon [0,1] \to [0,1/2]$ which is identically zero in a neighborhood of zero and identicaly one in a neighborhood of $1/2$ and is a diffeomorphism from $\mu^{-1}(0,1/2) \to (0,1)$, and define

$$F_1 = F \circ (\mathrm{Id}_X \times \mu)\colon X \times [0,1/2] \to M$$

and

$$G_1 = G \circ (\mathrm{Id}_X \times \mu)\colon X \times [0,1/2] \to M.$$

Then the map $F_1 * G_1\colon X \times [0,1] \to M$ that agrees with F' on $X \times [0,1/2]$ and agrees with the composition

$$X \times [1/2,1] \to X \times [0,1/2] \xrightarrow{G_1} M,$$

where the first map is translation by $-1/2$ in the t-coordinate, is an isotopy from f to h. $\square$

Definition 1.2. Suppose that $F\colon X\times[0,1]\to M$ is an isotopy. The *displayed version of* F is the map

$$\widetilde{F}\colon X\times[0,1]\to M\times[0,1]$$

defined by $\widetilde{F}(x,t)=(F(x,t),t)$. The displayed form of the isotopy is a level-preseving embedding of $X\times[0,1]\to M\times[0,1]$, where the level is measured by the t-coordinate.

The following is clear.

Lemma 1.3. *Given a level-preserving embedding* $\widetilde{F}\colon X\times[0,1]\to M\times[0,1]$ *we define* $F\colon X\times[0,1]\to M$ *as* $\pi\circ\widetilde{F}$ *where* π *is the projection onto the first factor. Then* F *is an isotopy of* X *in* M*. This gives a homeomorphism between the space of isotopies of* X *in* M *and the space of level-preserving embeddings of* $X\times[0,1]\to M\times[0,1]$.

Definition 1.4. An *ambient isotopy* is an isotopy H of $\mathrm{Id}_M\colon M\to M$ with the property that for each t the map H_t is a diffeomorphism of M onto M.

Let $f\colon X\subset M$ be a smooth embedding. An isotopy F of f *comes from an ambient isotopy* if F is the restriction to $f(X)\times I$ of an ambient isotopy H of M.

1.1 Isotopy Extension

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Theorem 1.5. *Let* X *be a compact manifold possibly with boundary and let* $F\colon X\times I\to M$ *be an isotopy. Then* F *comes from an ambient isotopy of* M.

Proof. Let $\widetilde{F}\colon X\times I\to M\times I$ be the displayed form of the isotopy F. Then image under $D\widetilde{F}$ of the vector field $\partial/\partial t$ is a vector field along $\widetilde{F}(X\times I)\subset M\times I$. It is of the form $(\xi(F(x,t)),\partial/\partial t)$ with $\xi(F(x,t))$ a tangent vector to M at $F(x,t)$. Since $X\times I$ is compact, the vector field $\xi(F(x,t))$ extends to a vector field ξ_1 tangent to M on an open neighborhood U of $\widetilde{F}(X\times I)$ in $M\times I$. Choose a smooth function μ identically 1 on $\widetilde{F}(X\times I)$ with support in U and let $\widetilde{\xi}|_U=\mu\xi_1$. Extend $\widetilde{\xi}|_U$ by 0 outside U and call the result $\widetilde{\xi}$. Then $(\widetilde{\xi},\partial/\partial t)$ is a vector field on $M\times I$ extending $\widetilde{F}_*(\partial/\partial t)$ on $\widetilde{F}(X\times I)$. Since the t-component of this vector field is $\partial/\partial t$ and since its M-component is zero outside a compact subset of M, all flow lines from this vector field extend from $M\times\{0\}$ to $M\times\{1\}$. Thus, the vector field $(\widetilde{\xi},\partial/\partial t)$ on $M\times I$

generates an ambient isotopy of M whose restriction to $F_0(X)$ is the given isotopy F of X in M.

Notice that the displayed version of the ambient isotopy is supported in the neighborhood U. For example, if the isotopy of X is disjoint from a closed subset Z of M, then we can take the ambient isotopy to be the constant identity isotopy on Z . □

2 Sard's Theorem

Definition 2.1. Let $f\colon M \to N$ be a smooth map between smooth manifolds. A point $x \in M$ is said to be a *critical value* for f if there is $y \in f^{-1}(x)$ such that $Df_y\colon T_yN \to T_xM$ is not surjective.

Remark 2.2. If the dimension of N is less than the dimension of M then the critical values of f are exactly the points in the image of f.

Theorem 2.3. *Let $f\colon N \to M$ be a smooth map between smooth finite dimensional manifolds. Then the critical vlaues of f are a nowhere dense subset of M. In fact they are a G_δ, a countable intersection of open dense subsets of M.*

A reference for the proof of Sard's Theorem is §1 of Chapter II of [3].

Sard's theorem also holds for Banach manifolds.

Recall that if $f\colon X \to Y$ is a smooth map and $Z \subset Y$ is a smooth submanifold, we say that f is transverse to Y if for every $x \in f^{-1}Z$, $Df_x(T_xX) + T_{f((x)}Z = T_{f(x)}Y$. In this case $f^{-1}(Z) \subset X$ is a smooth submanifold whose normal bundle in X is identified by *df* with the pullback under f of the normal bundle of Z in Y.

Corollary 2.4. *Suppose that we have a smooth family of smooth maps $f_p\colon X \to Y$ parameterized by p in a smooth manifold $\mathcal{P}$ and a submanifold $Z \subset Y$. Let $\mathcal{E} = \mathcal{P} \times X$. the maps f_p fit together to give a smooth map $F\colon \mathcal{E} \to Y$. If F is transverse to Z, then for a dense subset of $p \in \mathcal{P}$ $f_x\colon X \to Y$ is transverse to Z.*

Proof. Since $F\colon \mathcal{E} \to Y$ is transverse to Z, the preimage $W = F^{-1}(Z)$ is a smooth submanifold of $\mathcal{E}$ whose normal space at any point maps isomorphically onto the normal space for Z in Y at the image point. The regular values for the projection $W \to \mathcal{P}$ are parameter values p for which $f_p\colon X \to Y$ is transverse to Z. □

3 Thom Transversality

We shall uses the following form of transversality due to R. Thom. See §4 of Chapter II of [3].

Definition 3.1. Let M and N be smooth manifolds and dimensions m and n, respectively. For any k denote by $J^k(M,N)$ the bundle of k-jets of maps fro M to N. It is a bundle over $J^0(M,N) = M \times N$ and, using local coordinates on M and N, centered at $m \in M$ and $n \in N$, the fiber at (m,n) is identified with Taylor series of order k for functions from $\mathbb{R}^m \to \mathbb{R}^n$ sending 0 to 0. The jet bundles have natural smooth structures and there are compatible restriction mappings $J^k(M,N) \to J^\ell(MN)$ for each $\ell < k$. For a smooth function $f\colon M \to N$ there is a smooth map $J^k(f)\colon M \to J^k(M,N)$ defined by setting $J^k(f)(m)$ equal to the k-jet of f at m.

Theorem 3.2. *Let M and N be smooth manifolds and suppose that $W \subset J^k(M,N)$ is a smooth submanifold. Then for an open dense G_δ set of $f \in C^\infty(M,N)$ with the property that the map $J^k(f)\colon M \to J^k(M,N)$ is transverse to W.*

If M and N are smooth manifolds with boundary, we define $C^\infty(M,N)$ to be the space of C^∞-functions f that send $\partial M \to \partial N$ and $\mathrm{int}(M) \to \mathrm{int}(N)$ and are transverse along ∂M in the sense that the image under df sends any inward pointing vector at any point of ∂M to an inward pointing vector to N at the image point. Then for any smooth properly embedded submanifold $W \subset J^k(M,N)$ there is a dense G_δ of functions $f \in C^\infty(M,N)$ such that $J^k(f)$ is transverse to W.

Consider W=$M \times X \subset M \times N = J^0(M,N)$, In this case Thom's transversality theorem is the usual statement that a dense G_δ of smooth maps $M \to N$ are transverse to $X \subset N$.

There is also the multi-jet version of Thom's theorem.

For a smooth manifold M let

$$M^{(s)} \subset M^s = \underbrace{M \times \cdots \times M}_{s-\text{times}}$$

be the set of points $(x_1,\ldots,x_s)$ with $x_i \neq x_j$ for all $i \neq j$. This is a smooth manifold. We have the source map $J^k(M,N))^s \to M^s$. Let $J^k_s(M,N)$ be the pre-image under the source map of $M^{(s)} \subset M^s$. Then $J^k_s(M,N)$ is space of k-multi-jets at s distinct points of M. Any smooth map $f\colon M \to N$ defines a smooth map $J^k_s(f)\colon M^{(s)} \to J^k_s(M,N)$.

Theorem 3.3. *(Multi-jet transversality) Given a submanifold* $W \subset J^k_s(M,N)$ *there is a dense* G_δ *of smooth functions* $f\colon M \to N$ *such that* $J^k_s(f)\colon M^{(s)} \to J^k_s(M,N)$ *is transverse to* W.

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