Lecture V: Morse Theory on the loop space

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1 Introduction

There are several versions of the free (i.e., unbased) loop space on a compact, smooth manifold. There is the purely topological version $\Lambda(M)$ consisting of all continuous maps $S^1 \to M$ with the compact open topology. Equivalently, fixing a Riemannian metric on M with associated metric d_M, one can define a distance function on $\Lambda(M)$ by

$$d(\gamma, \xi) = \max_{\theta \in S^1} d_M(\gamma(\theta), \xi(\theta)).$$

This metric defines the compact open topology. We could also consider $\Lambda^\infty(M)$ the space of C^∞ paths with the C^∞ topology. This is a Frechet manifold. Also, we can consider the Sobolev space $\Lambda^{(1,2)}(M)$ of absolutely continuous maps $S^1 \to M$ with 1 distributional derivative in L^2. This is a Hilbert manifold. The inclusions of the spaces $\Lambda^\infty(M) \subset \Lambda^{1,2}(M) \subset \Lambda(M)$ are homotopy equivalences.

Now we assume that M is connected. For many computations it is more important (and usually easier) to deal with the following context. Fix points $p, q \in M$ and let $\Omega(M, p, q)$ be the space of maps $\omega\colon [0,1] \to M$ with $\omega(0) = p$ and $\omega(1) = q$ with the compact-open (or equivalently C^0) topology. When $p = q$, this is the based loop space of M based at p, denoted $\Omega(M, p)$. Again there are various versions of these: the continuous version, the smooth version with the C^∞-topology and the Sobolev version $\Omega^{(1,2)}(M, p, q)$. These homotopy types of these space are independent of p, q and also the choice of topologies listed above.

Of course, we have the usual isomorphisms

$$\pi_k(M, p) \cong \pi_{k-1}(\Omega(M, p)), \quad \text{for all} \;\; k \geq 1.$$

2 Condition (C)

Let X be a Hilbert manifold with a Riemannian metric and let $E\colon X \to \mathbb{R}$ be a C^1-function. The Palais-Smale Condition (C) says that for any subset $S \subset X$ if

- there is a constant $C < \infty$ such that $|E(s)| \leq C$ for all $s \in S$, and
- the closure in $\mathbb{R}$ of $\cup_{s\in S}|\nabla E(s)|$ contains zero,

then there is a sequence $x_n \in S$ with a limit in X that is a critical point of E. Under this condition, for any constant C, the intersection of the critical points of E with the subset where $|E| \leq C$ is compact. The Palais-Smale condition allows us to do Morse theory in the infinite dimensional contexts to obtain results about homotopy and homology.

Theorem 2.1. *(Palais-Smale) The two main facts of finite dimensional Morse theory carry over to the infinite dimensional setting under this hypothesis. Namely, suppose that M is a Riemannian manifold of class C^3 and $E\colon M \to \mathbb{R}$ is a C^3-function satisfying Condition (C). Then for any $-\infty < a < b < \infty$,*

- *if there are no critical points of E with critical value in $[a, b]$ then $E^{-1}([a, b]$ is diffeomorphic to $E^{-1}(a) \times [a, b]$, and*
- *if E is Bott-Morse then the critical submanifold in $E^{-1}([a, b])$ is a compact submanifold Σ and $E^{-1}([a, b])$ is diffeomorphic to $E^{-1}(a) \times [a, b]$ union the descending manifold $D_-(\Sigma)$ attached to $E^{-1}(a) \times \{b\}$ interection along the intersection $S_-(\Sigma) = D_-(\Sigma) \cap E^{-1}(a)$.*
- *If E has only non-degenerate critical points, then there are only finitely many ciritical points in $E^{-1}([a, b])$ and $E^{-1}([a, b])$ is diffeomorphic to $E^{-1}(a) \times ([a, b]$ union a finite number of handles, the descending manifolds of the critical points attached along the descending spheres at level a.*

Proof. (Sketch) If there is no critical point in $E^{-1}([a, b])$, then Condition (C) implies that $|\nabla E|$ is bounded away from 0 on $E^{-1}([a, b])$. Given $x \in E^{-1}([a, b])$ the flow line $\xi(t)$ for ∇f with $\xi(0) = x$ in either is defined for all $t \in [0, \infty)$ or meets $E^{-1}(b)$. Since $|\nabla E|$ is bounded away from 0 on $E^{-1}([a, b])$, it follows that $(E \circ \xi)'(t) = (\nabla E(\xi(t)))^2$ is bounded away from zero and hence there is some $t_+ > 0$ such that $E(\xi(t_+)) = b$. The symmetric argument shows that there is $t_- < 0$ such that $E(\xi(t_-)) = a$. Thus, all flow lines for ∇E go from $E^{-1}(a)$ to $E^{-1}(b)$. The usual argument shows that the map $E^{-1}(a) \times [a,b] \to E^{-1}([a,b])$ given by integrating flow lines is a diffeomorphism.

It follows easily from Condition (C) and the intersection of the critical set of E with $E^{-1}([a,b])$ is compact. Under the Morse-Bott condition it is a compact finite dimensional manifold (meaning finitely many components, each a compact, finite dimensional smooth manifold.) The local Morse picture of ascending and descending bundles from a Morse-Bott manifold of singularities carries over easily to the case of infinite dimensional Riemannian manifolds. □

2.1 The case of the energy functional on the loop space

Fix a finite dimensional Riemannian manifold M and points $p, q \in M$. Denote by $\Omega^{(1,2)}(M,p,q)$ the space of absolutely continuous paths $\gamma\colon [0,1] \to M$ from p to q with $\dot{\gamma}$ in L^2. It tangent space at γ is the space of L^2 vector fields along γ vanishing at the endpoints. Define the energy functional $E\colon \Omega^{(1,2)}(M,p,q) \to \mathbb{R}$ by

$$E(\gamma) = \frac{1}{2}\int_0^1 ||\dot{\gamma}(t)||^2 dt.$$

This is a smooth function. The gradient $\nabla E(\gamma)$ is $\dot{\gamma}(t)$ in the tangent space to the space of paths at γ. Let us sketch the proof that the Palais Condition (C) holds for the energy functional $E\colon \Omega^{(1,2)}(M,p,q) \to \mathbb{R}$. The point is that for any path $\gamma \in E^{-1}([0,c])$ and any $t_0 < t_1$ by Cauchy-Schwarz we have

$$d(\gamma(t_0), \gamma(t_1)) \leq \sqrt{c(t_1 - t_0)}.$$

Now suppose that we have a sequence of paths γ_n in $E^{-1}([0,c])$ with $|\nabla E(\gamma_n)| \mapsto 0$. This sequence of paths forms an equi-continuous family and thus there is a limit path γ_∞ that is a continuous path and the convergence is uniform in the C^0-topology. To complete the proof that the limit is a critical point for E we need to show that γ_∞ is path in $\Omega^{1,2}$, i.e., that it has a derivative in L^2. For it that is the case then since ∇E is continuous it will vanish on γ_∞. The argument that γ_∞ has an L^2 derivative is a classical, but slightly involved, argument that I will not explain. For a proof see §13 and §14 of 'Morse Theory on Hilbert Manifolds' by Richard Palais.

3 Sample Applications to the Free Loop space

Fix a compact Riemannian manifold. We consider the free loop space $\Lambda^{(1,2)}(M)$, the maps $\gamma\colon [0,1] \to M$ with its energy functional

$$E(\gamma) = \int_0^1 ||\dot{\gamma}||^2 dt.$$

The tangent space $\Lambda^{(1,2)}(M)$ to γ is $L^{(1,2)}$ vector fields along γ.

A loop γ is a critical point for E if for any one-parameter family $\gamma(t,s)$ of loops with $\gamma(t,0) = \gamma$ we have $dE/ds = 0$. Let $\eta(t)$ be a tangent vector field tangent t along γ, and suppose that the partial derivative in the s direction of a family $\gamma(t,s)$ at $s = 0$ is $\eta(t)$. Then for γ to be a critical point, we must have

$$dE(s)/ds|_{s=0} = 2\int_0^1 \langle \nabla_{\eta(t)}\dot{\gamma}(t), \dot{\gamma}(t)\rangle.$$

Since the connection is the Riemannian connection it is torsion free so that $\nabla_{\eta(t)}\dot{\gamma}(t) = \nabla_{\gamma(t)}\eta(t)$. Making this substitution and integrating by parts give

$$dE/ds = -2\int_0^1 \langle \eta(t), \nabla_{\dot{\gamma}(t)}\dot{\gamma}(t)\rangle dt.$$

It follows that this expression vanishes for all tangent vectors $\eta(t)$ if and only if $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all t. That is to say the critical points of E are the closed geodesics (meaning also that the parametrization is at the constant speed which is the length of the geodesic).

In this situation E is invariant under the action of the orthogonal group $O(2)$ acting on the parameter of the domain. Thus, the geodesics are not isolated. The best we can hope for is that the space of geodesics is a finite dimensional manifold. More speciafically we can hope that E is a Bott-Morse function.

Recall that we say that a function f is *Bott-Morse*, if each component of the set of its critical points is a smooth submanifold and at each point the kernel of the Hessian is equal to the tangent space to the critical set at that point submanifold. It is easy to see at any minimum the kernel of the Hessian is the tangent space to the copy of M consisting of trivial loops, so that this component satisfies the Bott-Morse condition. A standard application of Sard's Theorem shows that for a generic metric the energy functional is Bott-Morse and the non-trivial critical points are a discrete union of $O(2)$-orbits.

By the Palais-Smale condition, for any $A < \infty$. the intersection of the critical set with $E \leq A$ is compact, and for $\epsilon > 0$ sufficiently small, there are no non-trivial critical points on $E^{-1}([0, \epsilon))$. Also, the flow of $-\nabla E$ is defined for all $t \geq 0$. Finally, if there are no critical values in the interval $[s, t]$ then the gradient flow of $-\nabla E$ carries every point of $E^{-1}([0, t])$ into $E^{-1}([0, s])$. Thus, a slight generalization of ordinary Morse theory applies to this situation for a generic metric.

We define the *index of a geodesic* to be the dimension of the negative subspace of the Hessian. It turns out that the index is always finite. Assuming the critical point is of Bott-Morse type, the descending manifold for the $O(2)$-orbit is $D^r \times O(2)$ where r is the index. We form a chain complex by associating to each critical $O(2)\gamma$ a chain group isomorphic to $H_*(D(\gamma), \partial D(\gamma)$ where $D(\gamma)$ is the descending manifold near $O(2)\gamma$. Computing intersections in intermediate level sets as in the finite dimensional case leads to a boundary map for the direct sum of the chain groups associated to each non-constant $O(2)$-orbit of geodesics. This chain complex computes the relative homology $H_*(\Lambda(M), M)$ where $M \subset \Lambda(M))$ is the subspace of constant loops.

We say that a geodesic is *simple* if its stablizer under the $O(2)$-action is trivial. Each simple geodesic gives rise to an infinite family of geodesics mutiply covering it, one for each $n \geq 1$. Bott studied how the indices of the critical points of the multiple covers grow with n, and in particular showed that asymptotically they grow at least linearly with n. All of these results are related to establishing lower bounds, under various geometric and topological conditions, for the number of closed geodesics. Here are two sample results:

Theorem 3.1. *Let M be a closed manifold of positive dimension. Then M has a simple closed geodesic.*

Proof. If M has no non-trivial simple geodesics, then it has no non-trivial geodesics and hence E has no non-trivial critical points. Because of Condition C, it follows that the flow of $-\nabla E$ gives a deformation retraction of $\Lambda(M)$ to the space M of trivial critical points. This would imply that M and ΛM are homotopy equivalent. But the homotopy groups of $\pi_n(\Lambda M) \equiv \pi_n(M) \times \pi_{n-1}(M)$ where the first factor is the image of $\pi_n(M)$ under the inclusion of the trivial loops. Hence if M and ΛM are homotopy equivalent, $\pi_n(M) = 0$ for all n. But since M has positive dimension, it has non-trivial homology and hence non-trivial homotopy. □

More difficult arguments are required to prove:

Theorem 3.2. *(Gromoll-Meyer) Let M be a simply connected closed manifold. If the Betti numbers (with coefficients in some field of dharacteristic not equal to* 2*) grow without bound, then M has infinitely many distinct simple geodesics.*

Remark 3.3. Sullivan and Vigué-Poirrier and Sullivan showed that the hypothesis of the Gromoll-Meyer theorem is satisfied if the rational cohomology ring of M is not a truncated polynomial algebra, i.e., not generated by a single element.

Theorem 3.4. *(Klingenberg, Takens, Hingston, Rademacher,...) Let M be a closed simply connected manifold. Then for a generic C^4-metric M has infinitely many closed simple geodesics.*

4 Geodesics, Jacobi Fields, conjugate points, and the index

Now we turn to the based loop space, or its close cousin $\Omega^{(1,2)}(M,p,q)$. These is the space of paths parametrized by $[0,1]$ that are absolutely continuous with one derivative in L^2 starting at p and ending at q. As above the energy functional E is

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt.$$

The tangent space to $\Omega^{(1,2)}(M,p,q)_\gamma$ at an element γ is the space of L^2-vector fields along γ vanishing at the endpoints. A critical point for E is a path that satisfies

$$\int_0^1 \langle \eta(t), \dot\gamma(t)\rangle dt = 0$$

for all tangent vectors η at γ. Integrating by parts we see that

$$\int_0^1 \langle \eta, \nabla_{\dot\gamma(t)}\dot\gamma(t)\rangle = 0$$

for all tangent vectors η, which implies that $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. [The last two equations must of course be interpreted in the distributional sense.] A standard bootstrap argument then shows that γ is a smooth path and the equations hold as pointwise equations. Thus the critical points of E are smooth paths γ from p to q satisfying

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0,$$

i.e., geodesics.

4.1 Jacobi Fields and the Hessian of E

Now let us consider the kernel of the Hessian of E at a geodesic γ. Consider a two-parameter family $\gamma(t, s_1, s_2)$ of paths from p to q with $\gamma(t, 0, 0)$ being a geodesic. For $i = 1, 2$, let η_i be tangent vector $(\partial\gamma/\partial s_i)|_{s_1=s_2=0}$ along $\gamma(t, 0, 0)$. We compute

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s_2}\big|_{s_2=0}(E(\gamma(t, s_1, s_2)) &= \int_0^1 \langle \nabla_{s_2}\dot\gamma, \dot\gamma\rangle|_{s_2=0}dt = \int_0^1 \langle \nabla_{\dot\gamma}\eta_2, \dot\gamma\rangle|_{s_2=0}dt \\ &= -\int_0^1 \langle \eta_2, \nabla_{\dot\gamma}\dot\gamma\rangle|_{s_2=0}dt\end{aligned}$$

Now

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial s_1}\big|_{s_1=0}\Big(\frac{\partial}{\partial s_2}\big|_{s_2=0}\big(E(t, s_1, s_2)\big)\Big) = \\ &= -\int_0^1 \big(\langle \nabla_{s_1}\eta_2, \nabla_{\dot\gamma}\dot\gamma(t)\rangle\big|_{s_1=s_2=0}dt - \int_0^1 \langle \eta_2, \nabla_{s_1}\nabla_{\dot\gamma}\dot\gamma\rangle\big|_{s_1=s_2=0}dt,\end{aligned}$$

The integral of the first summand in the last line vanishes since $\gamma(t, 0, 0)$ is a geodesic. Also,

$$\nabla_{s_1}\nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\nabla_{s_1}\dot\gamma + R(\eta_1, \dot\gamma)\dot\gamma = \nabla_{\dot\gamma}\nabla_{\dot\gamma}\eta_1 + R(\eta_1, \dot\gamma)\dot\gamma,$$

where R is the Riemanni curvature operator:

$$R(\tau_1, \tau_2) = \nabla_{\tau_1}\nabla_{\tau_2} - \nabla_{\tau_2}\nabla_{\tau_1} - \nabla_{[\tau_1\tau_2]}.$$

Thus, we have established

$$\frac{1}{2}\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\big|_{s_1=s_2=0}\big(E(\gamma(t, s_1, s_2))\big). = -\int_0^1 \langle \eta_2, \ddot\eta_1 + R(\eta_1, \dot\gamma)\dot\gamma\rangle dt.$$

Thus, η_1 is in the kernel of the Hessian if and only if

$$\mathcal{J}(\eta) = -(\ddot\eta_1(t) + R(\eta_1(t), \dot\gamma(t))\dot\gamma(t)) = 0 \quad \text{for all} \quad t. \tag{4.1}$$

[Here, we have used $\dot\eta$ to mean $\nabla_{\dot\gamma}\eta$ and $\ddot\eta$ to mean $\nabla_{\dot\gamma}\nabla_{\dot\gamma}\eta$.]

This is the *Jacobi* equation. Vector fields along γ satisfying this equation are called *Jacobi fields.* These give the first order condition for a variation to be a family of geodesics. If there is a non-zero Jacobi field along γ vanishing at $p = \gamma(0)$ and at $p' = \gamma(t)$, then p and p' are said to be *conjugate along* γ.

Notice that the Jacobi operator $\mathcal{J}(\eta)$ is a liner operator that is a zeroth order perturbation of the linear operator $\eta \mapsto -\ddot\eta$. Since the zeroth order perturbation is a compact operator, the eigenvalues of $\mathcal{J}$ are shifted by a uniformly bounded amount from those of $-d^2/dt^2$, which are the non-negative integral multiples of $4\pi^2$. Thus, for each $C < \infty$ the operator $\mathcal{J}$ has only finitely many eigenvalues bounded above by C and all generalized eigenspaces are finite dimensional. For a Jacobi field to be a tangent vector to $\Omega^{(1,2)}(M,p,q)$ then it must vanish at p and $q meaning$that q is conjugate to p along γ.

Corollary 4.1. *If p and q are not conjugate along any geodesic of M then every geodesic from p to q is a non-degenreate critical point of $\Omega^{(1,2)}(M)$. More generally, if γ is a geodesic from p to q and p and q are conjugate along γ then the mull space of the Hessian of E at γ is the space of Jacobi fields vanishing at p and q. This space has dimension $< n$.*

Proof. The only thing in this corollary that we have not yet established is that the Jacobi fields along γ vanishing at p and q has dimension at most $n-1$. Since the Jacobi equation is second order linear, there is a $2n$ dimensional space of solutions and the coordinates are the value and derivative of the field at p. Since we are requiring the field vanish at p we are left with the n-dimensional space given by the derivative of the Jacobi field at p. But there is always one field vanishing at p that does not vanish at any other point, namelythe one vanishing at p and with first derivative $\dot{\gamma}(0)$ at p. By symmetry the term involving the curvature vanishes on this vector field and hence this vector field satisfies $-\ddot{\eta} = 0$ and thus it is a linearly growing multiple of $\gamma(t)$ along γ. □

It is easy to see that p and q are conjugate along some geodesic if and only if there is $v \in T_pM$ with $\exp_p(v) = q$ and $D_v\exp_p$ not an isomorphism. That is to say the points conjugate to p along some geodesic are the critical values of $\exp_p$. Consequently,

Corollary 4.2. *For any $p \in M$ there is a countable intersection of open dense subsets of M consisting of q not conjugate to p along any geodesic. In particular, the set of $q \in M$ not conjugate to a fixed $p \in M$ is a dense subset.*

Now we fix a non-conjugate pair of points p and q in M and consider $\Omega^{(1,2)}(M,p,q)$. By the above discussion all critical points of $E \colon \Omega^{1,2}(M,p,q) \to \mathbb{R}$ are non-degenerate and hence isolated. Because of the Palais Smale Condition C for any $A < \infty$ there are only finitely many geodesics from p to q with energy at most A.

4.2 Jacobi Fields and the index of a critical point

We now assume that p and q are points of M that are not not conjugate along any geodesic. It follows that every critical point of $E\colon \Omega^{(1,2)}(M,p,q) \to \mathbb{R}$ (i.e., geodesic from p to q) is non-degenerate. Our next task here is to identify the index of the Hessian of E at a geodesic γ. Let J_γ be the space (finite dimensional vector space) of Jacobi fields along γ vanishing at $p = \gamma(0)$. For each $t \in (0,1)$ let $J_\gamma(t) \subset J_\gamma$ be the subspace of Jacobi fields that also vanish at $\gamma(t)$.

Theorem 4.3. *Given γ, there are only finitely many $t \in (0,1]$ such that $J_\gamma(t)$ is non-zero, Of course,* $\mathrm{dim} J_\gamma(t) \le \mathrm{dim} J_\gamma = \dim M$. *The index of E at γ is the sum over $0 < t < 1$ of the dimension of $J_\gamma(t)$.*

For a proof of this see Milnor's notes on Morse Theory.

Here is an indication of how a non-zero element V in $J_{t_0}(\gamma)$ leads to a deformation of γ lowering the energy. Since the restriction of V to $[0,t_0]$, is a Jacobi field vanishing at both endpoints, $V|_{[0,t_0]}$ is in the kernel of the Hessian for $\gamma|_{[0,t_0]}$. Associated to V there is a one-parameter family of curves between $\xi_s(t)$ defined on $[0,t_0]$ with endpoints p and $\gamma(t)_0$ and with the variation of E at $s=0$ being zero to second order. We form a family $\gamma(t,s)$ by adjointing the constant curve $\gamma[t_0,1]$ to each $\xi(t,s)$. This is a family of curves for which the variation of E (without taking into account the break' at t_0) vanishes to second order at $s=0$. But these curves have a 'break' at t_0. The angle between $\xi(t,s)$ and $\gamma|_{t_0,1]}$ at t_0 has a non-zero first derivative in the s-direction. The reason is that V cannot vanish to second order at t_0, since it is non-zero and is a solution to a second order linear ODE. This means that by rounding the curves near the break, we form a new family of curves from p to q with second order variation of E being negative.

4.3 A finite dimensional model

Following Morse, Milnor works in a finite dimensional approximation to the subspace $\Omega(M,p,q)^c$ of the path space $\Omega(M,p,q)$ of consisting of paths of energy $< c$. Fix $\epsilon > 0$ with the property that any two points in M at distance less than ϵ apart are connected by a unique minimal geodesic that depends smoothly on the endpoints. The space $\mathcal{B}^c$ of broken geodesics defined on $[0,1]$ and broken at intervals of size $1/N$ where $(1/N) < \epsilon^2/c$ is identified with an open subset of $\prod_N M$. Under this identification the energy function E is a smooth function on $\mathcal{B}^c$. Following Morse, Milnor goes on to show that $\Omega(M,p,q)^c$ deformation retracts onto $\mathcal{B}^c$; that the critical points of E on $\Omega(M,p,q)^c$ are the same as the critical points of E on $\mathcal{B}^c$; and that the nullity and index of corresponding critical points in the two contexts are the same. This proves that $\Omega(M,p,q)^c$ is homotopy equivalent to a CW complex with one cell for each geodesic from p to q of length $< \sqrt{c}$, with the index of the cell equal to the index of E at the corresponding geodesic. In fact, one can take a limit as $c \mapsto \infty$ and show that $\Omega(M,p,q)$ is homotopy equivalent to a CW complex with one cell for each geodesic from p to q with the dimension of the cell being the index of the energy function at the critical point.

5 Floer Theory

There is another infinite dimensional context where there is a version of Morse theory. The first example of this was Floer's result proving many cases of the Arnold conjecture. We fix.a symplectic manifold (M,ω). Let $L \subset M$ be a Lagrangian and φ_t be a (time dependent) Hamiltonian flow with $L' = \varphi_1(L)$ being transverse to L. The Arnold Conjecture says that the number of points of intersection of L and L' is at least the sum of the Betti numbers of L. Floer's infinite dimensional approach to this question is to consider the space of paths $\gamma(t),\ 0 \leq t \leq 1$, beginning in L and ending in L' (with the technical property that $(\gamma(t)$ is homotopic relative to its endpoints to a curve μ where $\mu(t) \in \varphi_t(L)$. One works with paths in an appropriate Sobolev space of paths in order to have a Banach space of paths.

The critical points are constant paths at points of intersection of L and L', and the condition at a critical point be non-degenerate is that the corresponding point of intersection of L and L' be transverse. Unlike the situation for the energy functional on the loop space, both the positive and negative eigenspaces of the Hessian are infinite dimensional.. Thus, the criticial points are of infinite dimension and infinite codimension. In this context the gradient flow equation is replace by a non-linear elliptic equation. What replaces space of flow lines from one critical point q to to another p is a map

$$u\colon I \times \mathbb{R} \to M$$

satoisfying:

- $u(\{0\} \times \mathbb{R}) \subset L$ and $u(\{1\} \times \mathbb{R}) \subset L'$,
- with the paths $\omega_t = u|_{I\times\{t\}}$ tending to the constant path at p as $t \mapsto +\infty$ and to q as $t \mapsto -\infty$, and

- u is pseudo-holomorphic with respect to a fixed generic almost complex structure J on M compatible with the synplectic form, i.e., $\partial_J u = 0$.

Let $\mathcal{M}(p, q)$ be the moduli space of such critical points. Since the equation defining $\mathcal{M}(p, q)$ is an elliptic equation, its differential has a finite index, which depends only on p and q. There is no absolute grading (index) of the critical points, but $\dim(\mathcal{M}(p, q))$ is a relative index between the two critical points q and p so that one can fix a degree associated with each critical point well-defined up to an overall shift. Floer defines a chain complex by taking the chain groups to have $\mathbb{Z}/2\mathbb{Z}$ associated with each critical point. The boundary map ∂p will be a linear combination (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) of critical points q of relative index one lower than p. The coefficient of q in ∂p is defined by counting components of $\mathcal{M}(p, q)$. There is an $\mathbb{R}$-action on the space of solutions given precomposing with translations in the $\mathbb{R}$-direction of the domain. Thus, the boundary map is the number of points (in a transverse situation) in the quotient $\mathrm{mathcalM(q,p)}/\mathbb{R}$. A nice geometric argument using 2-dimesional moduli spaces shows that $\partial^2 - 0$. The result homology is the Floer homology of $(L, L'))$. To complete the argument Floer shows that the resulting homology groups are invariant under deformation of the time-dependent Hamiltonian flow, so the result is the same as for a small flow generated by a generic Hamiltonian, i.e. a Morse function. Here the fixed points are the critical points of the Morse function and the flow lines are the usual flow lines. Hence, in this case the Floer homology equals the usual topological homology. By invariance of the Floer homology the result follows.

5.1 Instanton Floer Theory and Seiberg=Witten Floer Homology

This basic idea has been reproduced in many other situations. In gauge theory of 3-manifolds, the configuration space is the space of connection on a principal bundle with compact group over a 3-manifold Σ. with the functional is the Chern-Simons functional

$$CS(A) = \int_\Sigma (dA + \frac{2}{3} A \wedge A \wedge A).$$

The variational equation becomes

$$\delta_{\delta A} CS(A) = 2F_A \wedge \delta A.$$

so that the critical points are exactly the flat connections. The 'flow lines' between two flat connections A_- and A_+ are anti-self-dual connections on $\Sigma \times \mathbb{R}$ that approach asymptotically A_- at $-\infty$ and A_+ at $+\infty$.

If one decomposes a closed 4-manifold M into two pieces M_+ and M_- along a 3-manifold T, then there are relative Donaldson polynomial invariants for M_+ and M_- taking values in the Instanton Floer homology of $\pm T$. The Donaldson invariants of M are the pairing between these relative invariants. (Reversing the orientation on a 3-manifold replaces its Instanton Floer with the dual.)

There is a version of Floer homology for 3-manifolds based on the Seiberg-Witten equations for 4-manifold. This is the Seiberg-Witten Floer homology, or the Monopole Floer homology. There is an analogous gluing theorem in this context as well. Let M be a closed 4-manifold and $T \subset M$ be a 3-manifold separating M into M_+ and M_-. Then there is a gluing theorem expresses the Seiberg-Witten invariants of M as the pairing relative SW invariants of M_+ and M_-, these taking values in the Monopole Floer homology of $\pm T$.