Problem Set 3 for Lie Groups: Fall 2022

September 26, 2022

Problem 1. Show that for the Lie group $S^1 \times \cdots \times S^1$ the exponental mapping is the universal covering. Let N be Lie group of upper triangular 3×3 matrices with 1's down the diagonal. Show that the exponential mapping from its Lie algebra to N is a global diffeomorphism.

Problem 2. Let $\{X_1, \ldots, X_k\}$ generate the Lie algebra L. Show that any element of L is a linear combination of brackets of the form

$$[Z_1, [Z_2, [Z_3, \cdots Z_n]]] \cdots],$$

where each Z_i is one of the X_j.

Problem 3. Show that the symmetric group Σ_n on n elements has a presentation

$$\langle T_1, \ldots, T_{n-1} \big| T_i T_i = \{1\}; T_i T_j = T_j T_i \text{ for } \; |i - j| > 1; T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \rangle.$$

Problem 4. A Lie Algebra N is *nilpotent* if for some k, all brackets of k elements of N are zero. A Lie algebra A is *abelian* if all brackets of elements of A are zero. An *ideal* I in a Lie algebra A is a linear subspace of A with the property that $[A, I] \subset I$. Show that if I is an ideal of A then it is a subalgebra of A and there is an induced Lie bracket on A/I. Conversely, show that if $\varphi \colon L \to L'$ is a map of Lie algebras then (i) the image of φ is a sub Lie algebra of L', (ii) the kernel of φ is an ideal of L and (iii) φ induces a Lie algebra isomorphism between the kernel of φ and the image of φ.

The *center* of a Lie algebra A is the set of $X \in A$ such that $[X, A] = 0$. Show that the center of A is an ideal of A. A subalgebra of A is said to be *central* if it is contained in the center of A. Show that if N is a nilpotent Lie algebra, then there is a filtration $0 = N_k \subset N_{k-1} \subset \cdots N_2 \subset N_1 = N$ such that each N_i is an ideal in N and N_i/N_{i-1} is central in N/N_{i-1}

Problem 5. Let L be the linear space of strictly upper triangular 3×3 matrices. Show that L is a sub Lie algebra of $\mathfrak{gl}(3,\mathbb{R})$ and that L is a nilpotent Lie algebra. Compute explicitly the exponential map on L. What is the subgroup of $GL(3,\mathbb{R})$ whose Lie algebra is L?

Problem 6. Show that if A is a Lie algebra then $[A, A]$ is an ideal, and more generally, defining $A_0 = A$ and $A_n = [A_{n-1}, A_{n-1}]$, each A_n is an ideal. A Lie algebra is *solvable* if $A_n = 0$ for some $n \geq 0$. Show each of the following is equivalent to A being solvable.

- There is a sequence of ideals $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = 0$ such that $[A_i, A_i] \subset A_{i+1}$.
- There is a sequence of subalgebras $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = 0$ with A_{i+1} an ideal in A_i such that $\dim(A_i/A_{i+1}) = 1$ for all i.
- $[A, A]$ is nilpotent.

Problem 7. Show that if B, C are solvable ideals of a Lie algebra A so is $B + C$. Deduce that a finite dimensional Lie algebra has a unique maximal solvable ideal, one containing all solvable ideals. This is the *radical* of A.

Problem 8. Consider the Lie group whose underlying manifold s $\mathbb{R}^+ \times \mathbb{R}$ with group action

$$(a', b') * (a, b) = (a'a, ab' + b).$$

Show that this defines a Lie group. Compute its Lie algebra and the exponential mapping.

Problem 9. Let $PSL(2,\mathbb{C})$ be the Lie group which is the quotient of $SL(2,\mathbb{C})$ by $\{\pm\mathrm{Id}\}$. What is its Lie algebra?

Problem 10. Let G be a connected Lie group and $T \subset G$ a closed, discrete subset of T that is also a sub Lie group. Show that T is an abelian subgroup and indeed is in the center of G..