Problem Set 3B for Lie Groups: Fall 2022

October 11, 2022

Problem 1. An *ideal* I in a Lie algebra $\mathfrak{g}$ is a linear subspace such that $X, I \subset I$ for every $X \in \mathfrak{g}$. Show that the quotient of a Lie algebra $\mathfrak{g}$ by an ideal I inherits a Lie bracket, making it a Lie algebra. Conversely, show that if $\rho \colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras, then $\mathrm{Ker}(\rho)$ is an ideal in $\mathfrak{g}$.

Problem 2. Show that if M and P are ideals of a Lie algebra G then so is $M + P$. Show that if M and P are nilpotent ideals then so is $M + P$. Show that G has a maximal nilpotent ideal, its *nilradical*.

Problem 3. A Lie algebra L is *solvable* if the series defined inductively by $L_1 = L$ and $L_n = [L_{n-1}, L_{n-1}]$ for all $n > 1$, satisfies $L_n = 0$ for some $n > 1$. Show the L_n are ideals and that if L is solvable then $[L, L]$ is a nilpotent ideal.

Problem 4. Let G be a connected Lie group and $\pi \colon M \to G$ a connected covering of G. Fix a point $m_0 \in \pi^{-1}(e) \subset M$. Show that there is a unique group structure on M with m_0 as the identity element and π a homomorphism. Show that this group structure together with the unique differential structure on M for which π is a local diffeomorphism is a Lie group and with this structure on M the map $\pi \colon M \to G$ is a homomorphism of Lie groups. Show that the $\mathrm{Ker}(\pi)$ is contained in the center of M and is a discrete subgroup.

Problem 5. Compute the center of $SL(n, \mathbb{C})$ for $n \geq 2$.

Problem 6. Compute the fundamental group of $PSL(2, \mathbb{R})$. Show that there is a covering group $G \to PSL(2, \mathbb{R})$ with the center of G being isomorphic to $\mathbb{Z}$.

Problem 7. Show any finite dimensional linear real representation of $SO(2)$ is completely reducible; i.e., a sum of irreducible representations. Show that an irreducible representation of $SO(2)$ is either the trivial one-dimesnsional representation or is isomorphic to the action of $SO(2)$ thought of as the unit circle in $\mathbb{C}$ acting on $\mathbb{C}$ (viewed as a two-dimensional real vector space) via $s \cdot w = z^k \cdot w$ for some $k \in \mathbb{Z}^+$.

Problem 9. Explicitly compute the Lie algebra of $SO(3)$. Show that it has no non-trivial ideals. Compute the Lie algebra for $SO(n)$, $n > 3$ and show it has no non-trivial ideals.

Problem 10. Show that $\mathfrak{s}l(n, \mathbb{R})$ is an ideal in $\mathfrak{g}l(n, \mathbb{R})$.

Problem 11. Let M be an n-dimensional manifold. A *flow box* for a k-dimensional folitation is a coordinate patch of the form $B^k \times B^{n-k}$ where B^r is the open unit ball in $\mathbb{R}^r$. The *local leaves* of the flow box are the submanifolds $B^k \times \{x\}$ for $x \in B^{n-k}$. Two flow boxes are *compatible* if at every point of their intersection the local leaves in the two flow boxes through this point have the same germ at the point. A *foliation* is determined by a compatible set of flow boxes covering M. The local flow boxes define a relation $x\widetilde{y}$ if x, y are in the same flow box and lie on the same local leaf. Extend this by transitivity to get an equivalence relation. Each equivalence class is a leaf of the foliation. A leaf of a foliation has two topologies: the subspace topology and the *leaf topology.* The latter is generated by the open sets in a local leaf in a flow box. Show that a leaf, with the leaf topology is a smooth manifold and is a one-to-one smoothly immersed k-dimensional submanifold M. Show that in general the subspace topology is different from the leaf topology.