Lie Groups: Answers to Final Exam
December, 16, 2025
1:10 pm - 4:00 pm

December 17, 2025

1. For $p, q \geq 0$, let $Q_{p,q}$ be the quadratic form on $\mathbb{R}^{p+q}$ given by

$$Q_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2).$$

Show that the orthogonal group of $Q_{p,q}$ is a Lie group. Show that this Lie group is compact if and only if either $p = 0$ or $q = 0$.

Solution. Let $O(p, q)$ denote the orthogonal group of $Q_{p,q}$. If p, q are both at least one, then $O(1, 1) \subset O(p, q)$ is a closed Lie subgroup. We take a different basis y_1, y_2 for $\mathbb{R}^2$ where the form is given by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (E.g., $y_1 = x_1 + x_2, y_2 = (x_1 - x_2)/2$) Then there is a proper embedding $\mathbb{R}^* \subset O(1, 1)$ defined by

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

This proves $O(p, q)$ is non-compact unless either $p = 0$ or $q = 0$

Changing the sign of a quadratic form does not change its orthogonal group so to complete the prolem it suffices to prove that the classical orthogonal group $O(n)$ is compact. But an element of $O(n)$ is determined by where it sends the standard unit basis vectors, $e_1, \ldots, e_n$. This determines a homeomorphismk between $O(n)$ and a subgroup of the n-fold product of the unit sphere with itself. The image is a set $v_1, \ldots, v_n$ of mutually orthogonal unit vectors and thus is a closed subset of the product of S^{n-1}'s and hence is compact.

2. Let L be a Lie algebra and let $x_1, \ldots, x_n \in L$ be elements. We define a *legitimate expression* in the $\{x_i\}_i$, by induction on the length of the expression. Any x_i is a legitimate expression of

length 1. If A and B are legitimate expressions of length k and ℓ, both at least 1, then $[A, B]$ is a legitimate expression of length $k+\ell$. Show that any legitimate expression of length r in $x_1, \ldots, x_n$ determines an element of L. Show any such element of L can be written is a linear combination with integer coefficients of a legitimate expressions of the form

$$[x_{i_1}, [x_{i_2}, \ldots, [x_{i_{r-1}}, x_{i_r}] \cdots].$$

Solution. We prove the first part by induction of the length of the expression. Each x_i is an element of L, so these represent elements of L. Suppose we have an expression $[A, B]$, with A and B each of length at least 1, of total degree k and we know the result for expressions of length $< k$. Then A and B represent elements of L and this expression represents the bracket of these element.

We call an expression of the type given at the end of the problem as being an expression of *standard form*. We prove the second part by double induction. The outer induction is on the length of the expression. The result is tautologically true for expressions of length 1 and 2. Suppose that for some $k \geq 2$, we know the result for expressions of length $< k$ and we have an expression of length k. By definition, any expression of length k is of the form $[A, B]$ with A and B of length at least 1. If the length of A is one then by induction, we can rewrite B as an integral linear combination of expressions of standard form. Since A is a single element, this gives the required rewritiing of $[A, B]$ as an integral linear combination of terms in standard form. Suppose now we know the result of all $[A, B]$ of total length k with the length of A less than ℓ, and suppose that A has length ℓ. Since $\ell < k$, we can write A as an integral linear combination of terms of the form $[a, A'_i]$, with a_i one of the x_j. Let examine each of these. Now by the Jacobi identity

$$[[a_i, A'_i], B] = -[[B, a_i], A'_i] - [[A'_i, B], a_i] = [A'_i, [B, a'_i]] + [a_i, [A'_i, B]].$$

Since the length of A'_i is less than the length of A_i and the length of $[A'_i, B]$ is less than $[A, B]$, the result follows by induction for $[A, B]$. This completes the inductive argument.

3. Let G be a compact group of rank 2. List the possible dimensions for G, and for each dimension, give the Cartan matrix and all fundamental groups, up to isomorphism, for compact Lie groups of rank 2 of that dimension.

Solution. Since G has rank two, the dimension of the root system (the dimension of the subspace of $\mathfrak{t}^*$ spanned by the roots) has dimension $0, 1$, or 2. If it is dimension 0, the G has dimension 2 and is a torus. Its Cartan matrix is empty and its fundamental group is isomorphism to $\mathbb{Z} \oplus \mathbb{Z}$.

If the root system has dimension 1, then component of the identity of the center of G is S^1 and the quotient by this S^1 is either $SU(2)$ or $SO(3)$. (The only rank one groups with roots.) Thus, the dimension of these groups is 4, the possible fundamental groups are, up to isommorphism, $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The Cartan matrix is the one-by-one matrix (2).

If the dimension of the root system is 2, then the quotient of the dual to the root system over the co-root lattice is a finite group. Thus, all the compact groups we are considering have finite fundamental groups. We have classified with 2-dimensional root systems: They are A_2, B_2, G_2. A_2 is the simply laced group. B_2 has long and short roots whose lengths squared differ by a factor of 2 and G_2 has long and short roots whose lengths squared differ by a factor of 3. In the three cases the Cartan matrices are

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Now the Cartan matrix is the pairing $\langle \alpha_i \alpha_j^\vee \rangle$, which is the pairing between R^* the root lattice and Λ_0. On the other hand, the quotient of R^*/Λ_0 is the center of the simply connected form of the group. One sees easily that the determinant of the Cartan matrix in the three cases is $3, 2, 1$. Thus, the centers of the simply connected forms are $\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$, and $\{1\}$ in the three cases. In addition, we have the adjoint form as well with fundamental groups $\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \{1\}$, respectively. There are no other possibilities in these three cases. A_2 has 6 roots and hence dimension of the associated groups is 8; B_2 has 8 roots and the dimension of the associated groups is 10 and G_2 has 12 roots (6 short and 6 long, each making up an A_2), so G_2 has dimension 14.

4. Let $\mathfrak{g}$ be a semi-simple real Lie algebra. Without appealing to Ato's theorem show that there is a simply connected Lie group whose Lie algebra is isomorphism to $\mathfrak{g}$. Show that up to isomorphism there is only one such group.

Solution. Since $\mathfrak{g}$ is semi-simple, the adjoint representation is injective. Hence $\mathfrak{g}$ is isomorphic to a sub Lie algebra of $\mathfrak{gl}(\mathfrak{g})$. By Lie's theorem there is a connected Lie group G and a morphism $G \to GL(\mathfrak{g})$ whose differential at the identity identifies the Lie algebra of G with $\mathfrak{g}$. Passing to the universal covering $\widetilde{G}$ of G gives us a simply connected Lie group with Lie algebra identified with $\mathfrak{g}$.

If $\widetilde{G}'$ is another such simply connected Lie group, Lie's second theorem produces a map of Lie groups $\widetilde{G} \to \widetilde{G}'$ that gives the identification of their Lie algebras. This is a covering map, but since $\widetilde{G}'$ is simply connected and $\widetilde{G}$ is connected, the map is a one-sheeted covering, i.e., a diffeomorphism.

5. Let R be an indecomposable (meaning not an orthogonal direct summand of two non-empty root systems). Show that if R is simply laced then all roots are conjugate under the Weyl group and there is a unique root in the closure of the fundamental domain. If R is not simply laced show that there are exactly two lengths of roots of R, which we call *long* and *short*. Show all long roots of R are conjugate under the action of the Weyl group and similarly for all short roots. Show there is a unique long root in the closure of the fundamental chamber and similarly for the short roots.

Solution. The Simply Laced Case. Take a Weyl invariant metric on the root space V, Suppose that U is a Weyl-invariant subspace of V. Then $V = U \oplus U^{\perp}$. Since reflection in any root α leaves this decomposition invariant either $\alpha \in U$ or $\alpha \in U^{\perp}$. Since, by assumption, we cannot divide the roots into two non-empty subsets $R_1 \coprod R_2$ with every root in R_1 orthogonal to every root in R_2, either $U = V$ or $U = \emptyset$. This implies that the orbit of any root spans V. That is to say, for roots α and β, there is an element $w \in W$ with $(w \cdot \alpha, \beta) \neq 0$. Either $\alpha = \pm\beta$, or $\{w \cdot \alpha, \beta\}$ generate a subroot system A_2 inside R. In both cases and hence $w \cdot \alpha$ and β are conjugate by an element of the Weyl group generated by reflections in α, β. It follows that α and β are in the same Weyl orbit, or said another way, all roots are conjugate under the Weyl-action.

Choose a notion of positive roots and let C_0 be the fundamental Weyl chamber, with its associated notion of simple roots $\{\alpha_1, \ldots, \alpha_k\}$. Begin with $r_1 = \alpha_1$. If it is possible to add a simple root to α_1, and get a positive root r_2 choose one and add it, forming a positive root r_2. Continue in this way as long as possible, constructing positive roots r_j with $r_j = r_{j-1} + \alpha_{i_j}$. This process must terminate since the simple roots are linearly independent and there are only finitely many positive roots. Let d be the root at which this process terminates. Then, for every simple root α_i we have $(d, \alpha_i) \geq 0$, for if it were negative we could add α_i to d to form a new positive root. This means that $d \in \overline{C}_0$.

Since all roots are conjugate under the Weyl action and since the quotient of the Weyl action on $\mathfrak{t}^*$ is identified with the closure of C_0, every W-orbit meets the closure of C_0 in a single point. Since the roots are an orbit, d is the only root in the closure of C_0.

The Non-Simply Laced Case. The argument above shows that the orbit of any root spans the root span. Thus, given any two roots α and β, there is $w \in W$ such that $(w \cdot \alpha, \beta) \neq 0$. Thus, the ratio of the squares of the lengths of α and β differ by a factor of $1, 2$, or 3. But if we had three different lengths $\{\ell_1, \ell_2, \ell_3\}$ each pair of lengths would have to satisfy this condition and that is not possible. Thus, there are only two lengths of roots.

The argument in the simply laced case now works equally well here to show that the long roots form a W-obit and the short roots for a W orbit, and that there is exactly one long root and one short root in the closure of C_0.

6. Let S be a set and let $L(S)$ be the free Lie algebra generated by S. Let $U(L(S))$ be the universal enveloping algebra of $L(S)$. Define inverse isomorphisms of algebras between $U(L(S))$ and the tensor algebra, $T(S)$, on S and prove that they are inverses of each other.

Soultion. The $T(S)$ is the free associative algebra generated by S. We have an inclusion $S \to U(L(S))$ with $U(L(S))$ being an associative algebra. Thus, there is a unique extension of this inclusion to an algebra homomorphism $T(S) \to U(L(S))$ extending the identity on S.

Conversely, $L(S)$ is the free Lie algebra on S and $T(S)$ is a Lie algebra with the $ab - ba$ bracket. We an the inclusion $S \to T(S)$. Since $L(S)$ is a free Lie algebra, this map extends to a map of Lie algebras $L(S) \to T(S)$. Now using the universal property of $U(L(S))$ we extend this to a map $U(L(S)) \to T(S)$ extending the identity on S. The compositions in the two orders give maps $T(S) \to T(S)$ and $U(L(S)) \to U(L(S))$ that are the identity on S. Since S generates both algebras (or by the uniqueness property) it follows that each of these compositions is the identity, so we have constructed inverse algebra isomorphisms between $T(S)$ and $U(L(S))$.

7. Let G be a Lie group with the property each element $g \in G$ is contained in a torus subgroup $T(g) \subset G$. Show that G is compact.[1]

Solution. Let $L \subset \mathfrak{g}$ be a maximal abelian sub algebra. Using Lie's theorem to integrate this gives a map $A \to G$ whose image is a closed, connected abelian group and whose differential at the origin is L. (If it weren't closed its closure would be an abelian subgroup with a larger Lie algebra.) That is to say, A is a closed Lie subgroup of G. It is of the form $T^k \times \mathbb{R}^\ell$.

Let us show $\ell = 0$. For, if $\ell > 0$, then there is an injective homomorphism $\psi\colon \mathbb{R} \to G$ whose image is a closed sub Lie group. Take a non-zero element

[1] This was the hardest problem; I didn't expect anyone to get it.

$g = \psi(t)$. By hypothesis, it is contained in a compact torus T in G. The same is true of all its powers. But the intersection of T with $\psi(\mathbb{R})$ is a compact subset of $\psi(\mathbb{R})$. This is a contradiction.

Thus, any maximal abelian sub algebra $\mathfrak{t}$ of $\mathfrak{g}$ integrates to a torus $T \subset G$ that is "maximal" in the sense that it is not contained in a larger torus. Now take one of these "maximal" tori, T. Since the Lie algebra of Z_0, the component of the identity of the center, is contained in any maximal abelian sub algebra of $\mathfrak{g}$, $Z_0$is contained in T. Of course, Z_0 is a closed subgroup of G. Since it is contained in a torus in G, it follows that Z_0 is compact.

We consider the adjoint representation of $T \times \mathfrak{g} \to \mathfrak{g}$. There is the trivial subspace of this action and 2-dimensional root spaces. Since T is not contained in a larger torus, the trivial subspace is the tangent space to the torus. The other root spaces are two dimensional and the roots associated with any of these subspaces are $\pm i\alpha$ where $\alpha\colon \mathfrak{t} \to \mathbb{R}$ and sends the lattice of T to $2\pi\mathbb{Z}$. Thus, the Killing form for $\mathfrak{g}$ restricted to $\mathfrak{t}$ is given by $B(H,H) = -2\sum_\alpha \alpha(H)^2$. This pairing is negative semi-definite on T with kernel equal to the intersection of the kernels of all the roots. Just as in the case of compact groups, this intersection is the central sub algebra of $\mathfrak{g}$, which is Lie algebra of $Z_0 \subset T$).

Now consider G/Z_0. The argument above shows that for each "maximal" torus T, the Killing form for the Lie algebra of G/Z_0 restricted to the Lie algebra T/Z_0 is negative definite. Since that is true for every "maximal" torus, and since by hypothesis these cover G and hence the quotients T/Z_0 cover G/Z_0, we see that the Killing form for the Lie algebra of G/Z_0 is negative definite. Consequently, G/Z_0 is compact. Since Z_0 is also compact, so is G.

8. Let R be a simply laced, indecomposable root system. Show that there is a unique root d which has the property that $d + \alpha$ is not a root for any simple root α. Show that the affine Weyl chamber contained in the fundamental Weyl chamber whose closure contains the origin has wall given by the vanishing of the simple roots and the wall given by $\{d = 2\pi\}$.

Solution. Fix a notion of positive roots. We saw in Problem 5 that there is a unique root d contained in the closure of the fundamental Weyl chamber $\overline{C}_0$. Every other positive root r is not in this chamber and hence there is a simple root α such that $(r, \alpha) < 0$. Then $r + \alpha$ is a root. We begin with the positive root r and continuing to add simple roots to create new positive roots until the process terminates. As in Problem 5, when the process terminates we have a root in the closure of C_0. But there is only one such root, namely d. This shows that for any positive root r, there is a collection of simple roots $\{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$ such that $d = r + \sum_j \alpha_{i_j}$.

Now we consider the intersection of C_0 with $\{d < 2\pi\}$. That is to say, we truncate C_0 by adding a new affine wall $\{d = 2\pi\}$ and consider the component of the complement of this wall in C_0 that contains the origin in its closure. This we claim is an affine Weyl chamber. If not then there is a root β and an integer k such that the wall $\{\beta = 2\pi k\}$ contains a point $x \in C_0$ with $d(x) < 2\pi$. If necessary, changing the sign of β and k, allows us to assume that β is a positive root. Since positive roots are positive on C_0 it follows that k is also positive. We have just shown that we have $\beta + \sum_j \alpha_{i_j} = d$, where the α_{i_j} are simple roots. Then we have $2\pi k + \sum_j \alpha_{i_j}(x) = d(x) < 2\pi$. Since the α_{i_j} are simple roots, they take positive values on x and k is a positive integer. This is a contradiction.