Problem Set 5 for Lie Groups: Fall 2025

September 26, 2025

Problem 1. Let $\{X_1, \ldots, X_n\}$ be generators for a Lie Algebra L. Show that every element of L can be written as a finite linear combination of elements of the form

$$[Z_1, [Z_2, [Z_3 \cdots [Z_{k-1}, Z_k]] \cdots]$$

where each Z_j is one of the X_i.

Problem 2. Let $H(X,Y) = H_1(X,Y) + H_2(X,Y) + H_3(X,Y) + \text{h.o.t}$ be the power series expansion for the Hausdorff series. Thus,

$$(1 + X + \frac{X^2}{2} + \frac{X^3}{3!} + \text{h.o.t})(1 + Y + \frac{Y^2}{2} + \frac{Y^3}{3!} + \text{h.o.t}) =$$
$$1 + (H_1 + H_2 + H_3) + (H_1 + H_2 + H_3)^2/2 + (H_1 + H_2 + H_3)^3/3! + \text{h.o.t}$$

Show that the Hausdorff series starts

$$H_1(X,Y)+H_2(X,Y)+H_3(X,Y) = (X+Y)+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]]).$$

Problem 3. Recall that a Lie algebra is nilpotent if there is an integer n such that all iterated brackets of length greater than n vanish. Prove that for any nilpotent Lie algebra N the Hausdorff series is a polynomial and that it determines the multiplication of a Lie group structure on N whose Lie algebra is canonically identified with N. Show exponential mapping, which we now view as a map from N with its Lie algebra structure to N with its Lie group structure is also a polynomial function.

For the next 3 problems we use the notation from the lecture: S is a finite set, $F(S)$ is the free associative algebra generated by S, $FL(S)$ is the free Lie algebra generated by S and $\hat{\psi}\colon U(F(S)) \to T(S)$ is the isomorphism between the universal enveloping algebra for $FL(S)$ to the tensor algebra of

S. All these algebras are graded algebras induced from the grading $S_\infty = \coprod_{n=1}^\infty S_n$. Using the restriction of the isomorphism $\hat{\psi}$ to $FL(S)$ deterimines an embedding of $FL(S)$ as. aub-Lie algebra of $T(S)$ with the Lie algebra structure derived from its natural associative algebra structure

Problem 4. Using the Lie algebra structure of $T(S)$ derived from its associative algebra structure, show that there is a map $P\colon T(S) \to T(S)$ defined by the linear extension of

$$P(s_1 \otimes s_2 \otimes \cdots \otimes s_n) = \mathrm{ad}(s_1) \circ \mathrm{ad}(s_2) \circ \cdots \circ \mathrm{ad}(s_{n-1})(s_n),$$

for $s_1, \ldots, s_n \in S$. Show that P preserves the gradings, has image contained in $FL(S)$, and is the identity on $S \subset T(S)$.

Problem 5. We keep the notation of P for the map defined in Problem 4. Define a map $\theta\colon T(S) \to \mathrm{End}(FL(S))$ by setting $\theta(s) = \mathrm{ad}(s)|_{FL(S)}$ for $s \in S$ and extending to all of $T(S)$ using the fact that $T(S)$ is the free associative algebra generated by S.
(a) Show that the restriction of θ to $FL(S)$ is a Lie algebra map from $FL(S)$ to $\mathrm{End}(FL(S)$ that sends $u \in FL(S)$ to the endomorphism $\theta(u) = \mathrm{ad}(u)\colon FL(S) \to FL(S)$.
(b) Show that for all $u \in FL(S))$ and all $v \in T(S)$ we have $P(uv) = \theta(u)P(v)$ for P the map in Problem 4.
(c) For $u, v \in FL(S)$ show that $P[u, v] = [P(u), v] + [u, P(v)]$.
(d) Show that for every k, the restriction of P to $FL^k(S) = T^k(S) \cap FL(S)$ is multiplication by k, so that the function $\overline{P}$ defined by

$$\overline{P} = \oplus_{k \geq 1} \frac{1}{k} P|_{T^k(S)} \colon T^k(S) \to FL^k(S)$$

is a projection $\overline{P}\colon T(S) \to FL(S)$ (meaning that $\pi|_{FL(S)} = \mathrm{Id}$).

Problem 6. Fix non-negative numbers m, r, s. Define $\alpha(m, r, s)$ to be the set of pairs of sequences $\mathbf{r} = (r_1, \ldots, r_m)$ and $\mathbf{s} = (s_1, \ldots, s_m)$ with the properties that (1) for each i we have $r_i + s_i \geq 1$ and (2) $|\mathbf{r}| := \sum_{i=1}^m r_i = r$ and $|\mathbf{s}| := \sum_{i=1}^m s_i = s$. Let $\alpha(m) = \cup_{(r,s)} \alpha(m, r, s)$. Show the following equality of formal power series:

$$\log(\exp(X)\exp(Y)) = \sum_m \frac{(-1)^{m-1}}{m} \left(\sum_{r,s \geq 0} \frac{X^r Y^s}{r!s!} \right)^m$$
$$= \sum_m \frac{(-1)^{m-1}}{m} \sum_{(\mathbf{r},\mathbf{s}) \in \alpha(m)} \frac{X^{r_1} Y^{s_1}}{r_1! s_1!} \frac{X^{r_2} Y^{s_2}}{r_2! s_2!} \cdots \frac{X^{r_m} Y^{s_m}}{r_m! s_m!}.$$

Problem 7. For $(\mathbf{r}, \mathbf{s}) \in \alpha(m)$: if $s_m \geq 1$ set

$$H(\mathbf{r}, \mathbf{s})(X, Y) = \frac{1}{|\mathbf{r}| + |\mathbf{s}|} \frac{\mathrm{ad}(X)^{r_i} \mathrm{ad}(Y)^{s_i}}{r_i! s_i!} \cdots \frac{\mathrm{ad}(X)^{r_m} \mathrm{ad}(Y)^{s_m - 1}}{r_m! s_m!}(Y),$$

and if $s_m = 0$ set

$$H(\mathbf{r}, \mathbf{s})(X, Y) = \frac{1}{|\mathbf{r}| + |\mathbf{s}|} \frac{\mathrm{ad}(X)^{r_i} \mathrm{ad}(Y)^{s_i}}{r_i! s_i!} \cdots \frac{\mathrm{ad}(X)^{r_m - 1}}{r_m!}(X),$$

(a) For all pairs of non-negative numbers (r, s) set

$$H(r, s)(X, Y) = \sum_m (-1)^{m-1} \frac{1}{m} \sum_{(\mathbf{r}, \mathbf{s}) \in \alpha(m, r, s)} H(\mathbf{r}, \mathbf{s})(X, Y).$$

Show that $H(r, s)$ is a finite sum of elements in $FL(S)$, each being a bracket of exactly r copies of X and s copies of Y. Show that we have an equality of formal power series.

$$\log(\exp(X)\exp(Y)) = \sum_{r,s \geq 0} H(r, s)(X, Y).$$

[Hint: Use the fact that $\log(\exp(X)\exp(Y)) \in FL(S)$.]

Problem 8. Consider the function $f(u, v) = -\log(2 - \exp(u + v))$. Show that the power series centered at $(u, v) = (0, 0)$ for f is

$$f(u, v) = \sum_{r,s \geq 0} \eta_{r,s} u^r v^s$$

where

$$\eta_{r,s} = \sum_m \frac{1}{m} \sum_{\alpha(m,r,s)} \frac{1}{r_1! s_1! \cdots r_m! s_m!}.$$

Show that this series is absolutely convergent for $u, v \geq 0$ and $u + v < \log(2)$.

Problem 9. Let L be finite dimensional real Lie algebra. Fix a positive definite inner product on L with associated norm denoted $|\cdot|$. Show that there is a $1 \leq M < \infty$ such that $|[U, V]| \leq M|U||V|$ for all $U, V \in L$. Show that for any fixed r, s and for elements $U, V \in L$

$$|H(r, s)(U, V)| \leq M^{r+s-1} |U|^r |V|^s \eta_{r,s}.$$

Problem 10. Show that with L a finite dimensional real Lie algebra and M as in the previous problem, the Hausdorff series $\sum_{r,s} H(r, s)(U, V)$ is absolutely convergent for U, V with $|U|, |V| < \frac{\log(2)}{2M}$.

Here are two elementary problems about completions of rings with respect to powers of an ideal.

Problem 11. Consider $\mathbb{Z}$ with the p-adic topology. By this we mean take the open sets of $0 \in \mathbb{Z}$ to be $(p)^n$ where (p) is the prime ideal generated by the prime number p. Show that the completion $\hat{\mathbb{Z}}_p$ of $\mathbb{Z}$ with respect to this topology is

$$\lim_{n \mapsto \infty} \mathbb{Z}/p^n\mathbb{Z}$$

with the maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ being the natural reductions. Show also that any element of $\hat{\mathbb{Z}}_p$ can be written as $\sum_{n=0}^{\infty} a_n p^n$, This expression is unique if we require in addition that all the a_n satisfy $0 \leq a_n < p$.

Problem 12. Let $R = k[x]$ and describe the completion of R with respect to the powers of the ideal (x) generated by x.