Problem Set 4 for Lie Groups: Fall 2025

September 20, 2025

Problem 1. Let L be the Lie algebra of strictly upper triangular real 3×3-matrices. Choose a basis and compute the Lie bracket explicitly in this basis. Show that L is the Lie algebra of the Lie group G of upper triangular real matrices with $1's$ down the diagonal. Compute explicitly the exponential mapping. Compute explicitly the $\exp(X)\exp(Y)$ where X and Y are general elements of L. Rewrite the answer as an exponential of a (finite) series of iterated brackets of copies of X and Y in L. Show this defines a group multiplication $L\times L\to L$. Show the exponential mapping is a Lie group isomorphism from this group structure on L to G.

Problem 2. Let K be a field of characteristic zero, show that any polynomial in $K[x^1,\dots,x^k]$ homogeneous of degree n is a finite linear combination of n^{th}-powers.

Problem 3. Let $\mathfrak{so}(3)$ denote the Lie algebra of real, skew-symmetric 3×3 matrices. Take the basis $X=A_{1,2}$, $Y=A_{2,3}$ and $Z=A_{1,3}$, where $A_{i,j}$ has 1 in the $\{i,j\}$ position, -1 in the $\{j,i\}$ position and zero elsewhere. Compute the brackets of these basis elements.

Problem 4. The tangent space of $SU(2)$ consists of matrices of trace 0 that staisfied $\overline{A}^t=-A$. Let X be the diagonal matrix with values $i,-i$ down the diagonal. Let Y be the matrix with entry 1 in position $\{1,2\}$ and -1 in position $\{2,1\}$ and let Z be the matrix with entry i in the $\{1,2\}$ and $\{2,1\}$ position. Show these are a basis for $\mathfrak{su}(2)$ and compute the brackets among these basis elements. Find an isomorphism between this Lie algebra and the one from Problem 3.

Problem 5. Let L be a Lie algebra with vector space basis $X_1,\dots,X_k$. Define $c^r_{i,j}$ so that $[X_i,X_j]=\sum_{r=1}^k c^r_{i,j}X_r$. Write the skew symmetry and Jacobi identity in terms of equations among the $c^r_{i,j}$.

Problem 6. Let L be a finite dimensional abelian Lie algebra. Pick a basis $\{X_1, \ldots, X_n\}$ and order the basis in the way indicated by the subscripts. Compute the map $U(L) \to P(L)$ given in the Lecture 4 Notes. Show it is an isomorphism of commutative algebras.

Problem 7. Let L be a 3-dimensional Lie algebra with basis X, Y, Z and brackets $[X, Y] = Z$ and $[Z, X] = [Z, Y] = 0$ Show that the Jacobi identity holds. Order the basis $X < Y < Z$ and determine explicitly the map $U(L) \to P(L)$ given by the construction in the Lecture 4 Notes. Show that that map is not an algebra homomorphism but the induced maps on the associated graded algebras is an algebra homomorphism. (Indeed, a graded algebra isomorphism.)