Lie Groups: Fall, 2025
Problem Set 3

September 18, 2025

These Problems present a guide for proving the Frobenius Theorem.

Problem 1. Give an example of a two-dimensional distribution in $\mathbb{R}^3$ that is not iinvolutive.

Problem 2a. Let $\mathcal{D}$ be a k-dimensional distribution on M^n (not necessarily involutive). For any $p \in M$ show there are neighborhoods $U_p \subset M$ of p, $V_x \subset \mathbb{R}^k$ of the origin, with coordinates $\overline{x} = (x^1, \ldots, x^k)$, and $U_y \subset \mathbb{R}^{n-k}$, with coordinates $\overline{y} = (y^1, \ldots, y^{n-k})$, of the origin and a diffeomorphism $\rho\colon U_x \times U_y \to U_p \subset M$ sending $(0,0)$ to p, such that the projection $\pi_x \circ \rho^{-1}\colon U_p \to V_x$ induces an isomorphism on each tangent plane of $\mathcal{D}|_{U_p}$.

Problem 2b. With the coordinates of Problem 2a, show that there are vector fields $X_1, \ldots, X_k$ on U_p tangent to $\mathcal{D}$ with $(\pi_x \circ \rho^{-1})_* X_i = \partial/\partial x^i$ for $1 \leq i \leq k$. Show these form a basis for the planes of $\mathcal{D}(y)$ for every $y \in U_p$. Show these coordinates are unique up to translations in some (possibly smaller) neighborhood of p.

Problem 2c. Now suppose that $\mathcal{D}$ is involutive. Let the X_i tangent to $\mathcal{D}_{U_p}$ be as in Part 2b. Show that $[X_i.X_j] = 0$ for all $1 \leq i, j \leq k$.

The point of Problem 3 is to prove by induction on k that given vector fields $X_1, \ldots, X_k$ in a neighborhood U of M that are (i) linearly independent at ever $q \in U$ and satisfy $[X_i, X_j] = 0$ for all $i, j \leq k$, then there is a smaller neighborhood $U' \subset U$ of p on which there are coordinates $(x^1, \ldots, x^n)$ such that $X_i|_{U'} = \partial/\partial x^i$ for all $1 \leq i \leq k$.

Problem 3a. Recall that if X be a vector field defined on a neighborhood U of a point $p \in M^n$. Then there is $\epsilon > 0$ and a unique smooth curve $\gamma\colon (-\epsilon, \epsilon) \to U$ with $\gamma(0) = p$ and $\gamma'(t) = X(\gamma(t))$. These curves vary smoothly with the initial condition p. Show that if W is a smooth submanifold of M with compact closure, then there is an $\epsilon > 0$ such that $\gamma_X(p)$ is defined on $(-\epsilon, \epsilon)$ for all $p \in W$ and the integral curves define a smooth map $W \times (-\epsilon, \epsilon) \to M$.

Problem 3b. Consider the case when $k = 1$, i.e., when there is only one (nowhere zero) vector field X_1 on U. Show that near every $p \in U$ there is a coordinate system $(x^1, \dots, x^n)$ in a neighborhood of p in which $X_1 = \partial/\partial x^1$.

Problem 3c. Suppose the result for $k' = k - 1$ and let $X_1, \dots X_k$ on a neighborhood U of $p \in M^n$ that (i) are linearly independent and (ii) satisfy $[X_i, X_j] = 0$ for all i, j. By induction we have coordinates $(x^1, \dots, x^n)$ near p such that $X_i = \partial/\partial x^i$ for $i = 1, \dots, (k-1)$. Show that that X_k is not contained in the span of $\{\partial/\partial x_0^i\}_{i \neq k}$. Show that we can assume that X_k is not in the span of $\{\partial/\partial x^i\}_{i \neq k}$. Then set U^{n-1} a small ball in $\{x^k = 0\}$. Show that integrating X_k over a small interval $(-\epsilon, \epsilon)$ determines an diffeomorphism $U^{n-1} \times (-\epsilon, \epsilon) \to V$ where V is an open subset of p. Let the t-direction in this product be the coordinate x^k on V and let x^j for $j \neq i$ be given by the projection $V \to U^{n-1}$ together with the coordinates on U^{n-1}. Show that $X_k = \partial/\partial x^k$. Using the fact that $[X_i, X_k] = 0$ show that $X_j = \partial/\partial x^j$ for $1 \leq j \leq k-1$ throughout V. Show that this completes the inductive argument.

Problem 4. Show that the tangent planes to a foliation form an involutive distribution.

Problem 5. Let X and Y be vector fields on a manifold M. We define the *Lie derivative*, $L_X(Y)$, of X on Y as follows. Fix $q \in M$. Let $\gamma_X(t)$ be the integral curve for X with $\gamma_X(0) = q$. Then

$$L_X(Y)(q) = \lim_{t \mapsto 0} \frac{\gamma_X(t)_*(Y(\gamma_X(t))) - Y(\gamma_X(0))}{t}.$$

Show that $L_X(Y) = [X, Y]$.

Problem 6. Let $\mathcal{F}$ be a foliation of M. Suppose that we have a covering of M by flow boxes $F_\alpha = U_\alpha \times V_\alpha$ with the U_α connected. Suppose in addition:

- the F_α are a locally finite covering of M
- each F_α has compact closure $\overline{F}_\alpha$ contained in an open flow box G_α.
- for any pair of flow boxes F_α and F_β and any local leaf of $U_\alpha \times \{y\}$ of F_α the closure of $U_\alpha \times \{y\}$ meets only one leaf in the closure of F_β.

Show that with these assumptions, every leaf of $\mathcal{F}$ meets each of the F_α in at most a countable set of local leaves. Show that the leaves of $\mathcal{F}$ are second countable.