Problem Set 2 for Lie Groups: Fall 2025

September 25, 2025

Problem 1. Let $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. Let $SL(2,\mathbb{R}) \times \mathbb{H} \to \mathbb{H}$ be the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Show that this action preserves the metric

$$\frac{ds^{\otimes 2}}{y^2}.$$

Show that $\pm 1 \subset SL(2,\mathbb{R})$ is the kernel of this action and the quotient group acts simply transitively on the space of unit tangent vectors of $\mathbb{H}$. Show that it is the group of orientation-preserving isometries of $\mathbb{H}$ with the metric given by the above formula.

Problem 2. Consider the complex manifold $\mathbb{C}^2 \setminus \{0\}$. The group $\mathbb{C}^*$ acts freely by scalar multiplication. Show that the quotient has the structure of a complex manifold and that it is covered by two coordinate patches, each isomorphic to $\mathbb{C}$, say one with coordinate z and the other with coordinate w, with the overlap being $w = z^{-1}$. Show the quotient space is diffeomorphic to S^2. We view S^2 as the one-point compactification of the z-plane by adding the point ∞ with a basis of the topology near ∞ consisting of all subsets of the form ∞ union the complement of the closed disk of radius $R < \infty$ about the origin in the z-plane.

Consider the usual action $SL(2,\mathbb{C}) \times \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}^2 \setminus \{0\}$. Show that this action descends to an action on S^2 that is by holomorphic isometries of S^2. Show that any orientation-preserving conformal isomorphism of S^2 that fixes ∞ is given by a complex linear map $z \mapsto az+b$ with $a, b \in \mathbb{C}$ and $a \neq 0$. Using this prove that every orientation-preserving conformal isomorphism of S^2 is the action of some element of $SL(2,\mathbb{C})$. Show that ± 1 acts trivially and that the quotient $PSL(2,\mathbb{C})$ is the group of orientation-preserving conformal isomorphisms of S^2.

Problem 3. Let G be a connected Lie group and $K \subset G$ a normal Lie subgroup. Show that if $\dim(K) = 0$, then K is contained in the center of G.

Problem 4. What is the subspace of space of $M(n \times n, \mathbb{R})$ tangent to the identity of $SO(n) \subset GL(n,\mathbb{R})$? Show that this subspace is closed under the $AB - BA$ Lie bracket on matrices.

Problem 5. Let ω be the standard non-degenerate skew-symmetric bilinear form on $\mathbb{R}^{2n}$, i.e., the one given by a block diagonal matrix whose blocks are

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Describe the Lie algebra of the group of linear transformations on $\mathbb{R}^{2n}$ that preserve this bilinear form.

Problem 6. Show that if X and Y are vector fields on a smooth manifold M, then $XY - YX$ is also a vector field.

Problem 7. Show that the space of right-invariant vector fields on a Lie group G is closed under bracket and is identified with T_eG. Show that the induced bracket on T_eG is the negative of the bracket induced by left-invariant vector fields. Show that the integral curve through the identity for a right-invariant vector field is a one-parameter subgroup and hence is the same as the integral curve through e for the left-invariant vector field with the same value at e.

Problem 8. Let B be an open ball centered at the origin in $\mathbb{R}^n$ and X and Y be vector fields defined on B. Then for all $\epsilon > 0$. Given $p \in B$ we define $\alpha_p(t)$ to be the integral curve of X through p and $\beta_p(t)$ the integral curve of Y though p. For $\epsilon > 0$ sufficiently small the following are defined for all $0 \le t < \epsilon$:

$$a_1(t) = \alpha_0(t), \quad a_2(t) = \beta_{a_1(t)}(t), \quad a_3(t) = \alpha_{a_2(t)}(-t), \quad a_4(t) = \beta_{a_3(t)}(-t).$$

Show that $a_4(t)$ is a smooth curve with $a_4(0) = 0$ and $a_4'(0) = 0$ and $a_4''(0) = [X,Y](0)$. [Hint: First consider the case when X and Y are linearly dependent at the origin. If they are linearly independent find coordinates on B near the origin such that $Y = \partial/\partial x^2$ and $X = \partial/\partial x^1$ along the hypersurface $x_2 = 0$.]

Problem 9. Let S^3 be the subspace of the quaternions of unit norm; i.e., $a + bi + cj + dk$ with $a^2 + b^2 + c^2 + d^2 = 1$. Give a vector space basis and all brackets of these basis elements for the Lie algebra of S^3. The same question for $\mathfrak{so}(3)$, the Lie algebra of the special rotation group $SO(3)$.

Problem 10. Let G be an abelian Lie group. Describe the Lie bracket on $\mathfrak{g}$.

Problem 11. Recall that a group N is *nilpotent* if it has a finite composition series by normal subgroups

$$\{e\} = N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_k = N$$

with the property that $[N, N_i] \subset N_{i-1}$, where the bracket here means the subgroup generated by all commutators of elements in N and N_i, i.e., for all i, the group N_i/N_{i-1} is contained in the center of N/N_{i-1}. Show that the Lie algebra of a nilpotent Lie group is a nilpotent Lie algebra in the sense that there is k such that all Lie brackets of length k or more vanish.

Problem 12. Show that a left-invariant Riemannian metric on G determines a Riemannian metric on the coset manifold $H\backslash G$ from the orthogonal splitting $T_g(G) = T_g(Hg) \oplus T_g(Hg)^{\perp}$ and the identification of the second factor with $T_{[g]}(H\backslash G)$. Show that if this metric is also right-invariant then the reulting action on $H\backslash G$ is invariant under the natural right action of G on $H\backslash G$.

Problem 13. Describe every element in $\mathfrak{sl}(2, \mathbb{R})$ up to conjugation. Show that any element in $SL(2, \mathbb{R})$ that is in the image of the exponential mappiing has trace ≥ -2. Show that $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is not onto.