Problem Set 1 for Lie Groups: Fall 2025

September 4, 2025

Problem 1. Fix a field K and consider the polynomial ring with variables X_{ij} with $1 \leq i, j \leq n$. This is the ring of polynomial functions (with coefficients in K) on K^{n^2}. We identify the space $M(n \times n, K)$ of $n \times n$ matrices over K with the vector space K^{n^2} in such a way that the function $X_{i,j}$ assigns to each matrix its (i, j)-entry. For any r, we give K^r the *Zariski* topology where the closed subsets are exactly the loci in K^r where some given collection of polynomial vanishes. (These are called *subvarieties.*) Show that this is indeed a topology. Show that $GL(n, K) \subset M(n \times n, K)$ is open in the Zariski topology. Show that $SL(n, K)$, the matrices of determinant 1, is a closed subset in the Zariski topology. Show that matrix multiplication is an algebraic map, meaning that it pulls back polynomial functions on $M(n \times n, K)$ to polynomial functions on $M(n \times n, K) \times M(n \times n, K)$. Show that this map is cotninuous in the Zariski topology. Show that $GL(n, K)$ and $SL(n, K)$ are groups under matrix multipliciation.

Problem 2. A *linear algebraic group over* K is an K-affine subvariety; i.e., a subvariety V of K^n for some n, and a algebraic morphisms $\mu\colon K \times K \to K$ and $\iota\colon K \to K$ together with a K-algebraic point $e \in V$ that satisfy all the group laws in the category of K-algebraic varieties and morphisms.. Show that $GL(n, K)$ with its usual matrix multiplication and inverses is a linear algebraic group.Show that multiplication and inverse are given by rational functions of the matrix entries where the denominator is a power Show that any subvariety of $GL(n, K)$ that is closed under multiplication and inverses and contains the identity matrix is a linear algebraic group over K. Show that any linear algebraic group over $\mathbb{R}$, resp. $\mathbb{C}$, is a Lie group, resp. a complex Lie Group.

Problem 3. Let Q be a positive definite quadratic form on a n-dimensional real vector space. Show that the orthogonal group of Q is a Lie group by showing that it is a smooth submanifold of $GL(n, \mathbb{R})$. [Hint: Show that there is no loss of generality in taking Q to be the standard Euclidean quadratic form. Then show a matrix $A \in M(n \times n, \mathbb{R})$ is in the orthogonal group if and only if its columns form an orthonormal basis of $\mathbb{R}^n$. Then use the implicit function theorem to show establish the result.] The result is the orthogonal group of Q, denoted $O(n)$.

Problem 4. Show that $O(n)$ has two components. Define $SO(n)$ to be the subgroup of orthogonal matrices of determinant 1. Show $SO(n)$ is the component of the identity of $O(n)$.

Problem 5. Given any non-degenerate quadratic form Q on $\mathbb{R}^n$ (meaning that if $Q(x+y) = Q(y)$ for all $y \in \mathbb{R}^n$, then $x = 0$). Show any such form can be diagonalized, i.e., there is a basis $e_1, \ldots, e_n$ such that $Q(e_i) = \pm 1$ and $Q(e_i + e_j) = Q(e_i) + Q(e_j)$ for all $i \neq j$. Define the orthogonal group of Q and show that it is a sub Lie group of $GL(n, \mathbb{R})$.

Problem 6. Show that the group of unitary $n \times n$-matrices, i.e., $A \in GL(n, \mathbb{C})$ satisfying $\overline{A}^{tr} = A^{-1}$ is a real Lie subgroup of $GL(n, \mathbb{C})$. Show that in general it is not a complex Lie subgroup.

Problem 7. Let A be a non-degenerate skew symmetric pairing on $\mathbb{R}^{2n}$. Define $Symp(2n, \mathbb{R})$ as the set of elements in $g \in GL(2n, \mathbb{R})$ that preserve A in the sense that $A(x, y) = A(gx, gy)$ for all $x, y \in \mathbb{R}^{2n}$. Show that $Symp(2n, \mathbb{R})$ is a sub-Lie Group of $GL(n, \mathbb{R})$.

Problem 8. Let $\mathbb{R}^+$ act on $\mathbb{R}^2 \setminus \{(0,0)\}$ by $t \cdot (x, y) = tx, t^{-1}y$. Show that this is a smooth action free action and every orbit is a closed submanifold of $\mathbb{R}^2 \setminus \{(0,0)\}$. Show the quotient space is not Hausdorff.

Problem 9. (Basic Topology) A topology on a set X is *second countable* if there is a countable basis for the topology (i.e., a countable collection of open sets $\{U_n\}_{n \in \mathbb{Z}}$ such that every open set is a union of a subset of the U_n). A topology is *first countable* if the topology at every point has a countable basis, meaning for each $x \in X$ there is a countable union of open subsets containing x such that any other subset contains an open set about x if and only if it contains one of the given collections. Show that a metric space is first countable. Show that a metric space is second countable if and only if it has a countable dense subset (dense meaning that the closure is the entire space).

Problem 10. Let G be a connected Lie group. Show that G is generated by any open neighborhood of the identity.