# Lie Groups: Fall, 2025
# Lecture IX:
# Root Systems

October 13, 2025

For this lecture we fix a compact, connected Lie group $G$, a maximal torus $T \subset G$.

## 1 'Lattices'

We have the action of the Weyl group $W$ on $T$ and the induced linear action of $W$ on $\mathfrak{t}$. We define the dual action $W \times \mathfrak{t}^* \to \mathfrak{t}^*$ by $w\varphi(X) = \varphi(\mathrm{ad}(w^{-1})X)$.

### 1.1 Basic Definitions

**Definition 1.1.** The basic lattice we have to begin with is $\Lambda \subset \mathfrak{t}$, the kernel of $\exp\colon \mathfrak{t} \to T$. It is identified both with the fundamental group of $T$ and with the lattice $\mathrm{Hom}(S^1, T)$. It is the *co-root lattice*.

The *weight space* is the lattice $\mathrm{Hom}(T, S^1)$. It is the lattice of all characters of all linear representations of $T$. It is identified with the lattice in $\mathfrak{t}^*$ of all linear maps $\mathfrak{t} \to \mathbb{R}$ that send $\Lambda \to 2\pi\mathbb{Z}$, and hence with $\mathrm{Hom}(\Lambda, frm{-}epi\mathbb{Z}) = \Lambda^*$. Clearly, the natural pairing $\mathfrak{t}^* \otimes \mathfrak{t} \to \mathbb{R}$ induces the usual dual pairing of the weight lattice and the co-root lattice, namely the tautological pairing $\Lambda^* \otimes \Lambda \to 2\pi\mathbb{Z}$.

**Definition 1.2.** The *root lattice* $R \subset \mathfrak{t}^*$ is the (possibly partial) lattice spanned by the roots of $(G, T)$. The root lattice is a subgroup of the weight lattice $\Lambda^*$ generated by the non-zero weights of the adjoint action of $T$ on $\mathfrak{g}$. The *co-weight space* is a subgroup of $\mathfrak{t}$ dual to the partial root lattice. By definition, it consists of all $v \in \mathfrak{t}$ with the property that $\alpha(v) \in 2\pi\mathbb{Z}$ for all $\alpha \in R$. In general, the root lattice is partial lattice in $\mathfrak{t}^*$ in the sense that is a discrete subgroup but imay not span $\mathfrak{t}^*$ over $\mathbb{R}$. We examine the structure of the co-weight space below.

Since the roots are characters of the adjoint action of $T$ on $\mathfrak{g}$, they are weights; i.e., $R \subset \Lambda^*$.

**Examples 1.** If $G$ is a torus $V/\Lambda$, then the co-root lattice is $\Lambda$ and the weight lattice is $\Lambda^*$. On the other hand, the root lattice is $\{0\}$ and the co-weight space is $\mathfrak{t}$.

**Example 2.** The root lattice is a (full) lattice if and only if the roots span $\mathfrak{t}^*$ over $\mathbb{R}$. This is the case exactly when the co-weight space is a lattice.

### 1.2 The Co-Weight Space and the Center of $G$

**Lemma 1.3.** *The image of co-weight space $\widetilde{Z} \subset \mathfrak{t}$ under the exponential map is the center of $G$. That is to say, the center of $G$ is the quotient of the co-weight space by the co-root lattice.*

*Proof.* By definition, the co-weight space is the intersection over all roots $\alpha\colon \mathfrak{t} \to \mathbb{R}$ of $\alpha^{-1}(2\pi\mathbb{Z})$. Thus, the image of the co-weight space under exponentiation is the intersection over all roots $\alpha\colon T \to S^1$ of the kernel of $\alpha$.

The adjoint action of $g \in T$ on $V_\alpha$ is the trivial action if and only if $\alpha(g) = 1$. Hence, the image of the co-weight space in $T$ is the kernel of the adjoint action of $T$ on $\mathfrak{g}$. These are exactly the elements in $T$ for which the adjoint action on $\mathfrak{g}$ is trivial. Since $G$ is connected the kernel of the adjoint action of $T$ on $\mathfrak{g}$ is equal to the kernel of the adjoint action of $T$ on $G$;i.e., all elements of $T$ contained in the center of $G$. On the other hand, we know the center of $G$ is contained in every maximal torus and hence is contained in $T$. □

**Corollary 1.4.** *The co-weight space in $\mathfrak{t}$ is the universal covering of a compact abelian group. Thus, it is isomorphic to $\mathbb{R}^k \times F$ where $F$ is a finite abelian group. The subgroup $\mathbb{R}^k$ of $\mathfrak{t}$ is the center of $\mathfrak{g}$; i.e., the kernel of* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$. *Say another way, it is the set of $X \in \mathfrak{g}$ such that $[X, \mathfrak{g}] = 0$.*

**Corollary 1.5.** *1. The root lattice is a full lattice if and only if the center of $G$ is finite if and only if the adjoint action of $\mathfrak{g}$ on itself is a faithful representation.*

*2. The root lattice equals the weight lattice if and only if the center of $G$ is the trivial group if and only if the adjoint action of $G$ on $\mathfrak{g}$ is a faithful representation.*

Often one only invokes the terms root lattice and co-weight lattice, in the case when these are (full) lattices; i.e., when the center of $G$ is finite.

**Definition 1.6.** A group with trivial center is said to be the *adjoint form*. For any compact group, its adjoint form is the quotient of $G$ by its center. The adjoint form of a Lie group $G$ is the image in $GL(\mathfrak{g})$ of the adjoint action of $G$ on $\mathfrak{g}$.

As an example, the adjoint form of $S^3$ is $SO(3)$. The adjoint form of a torus is the trivial group. (Often, one talks about the adjoint form only for groups with finite center, so that the adjoint form has the same Lie algebra as the original group.)

# 2 Action of the Weyl Group

## 2.1 Identification of $\mathfrak{t}$ and $\mathfrak{t}^*$

By the definitions, the roots $\alpha$ are elements of $\mathfrak{t}^*$ and the action of the Weyl group is on $\mathfrak{t}$. The generators of the Weyl group are the reflections in the Weyl walls $W_\alpha = \ker(\alpha)$ in the sense that they are the identity on $W_\alpha$ and interchange the sides of $W_\alpha$. As described above, we have the adjoint action of $W$ on $\mathfrak{t}^*$. Of course, the action on $\mathfrak{t}^*$ of the reflections in $W$ has the same structure as their action on $\mathfrak{t}$. That is to say, the action of a reflection on $\mathfrak{t}^*$ has a point-wise fixed codimension-one subspace. But we have no good description of the invariant subspace.

To get a much clearer picture, we fix a Weyl invariant metric on $\mathfrak{t}$, which we denote by $(\cdot,\cdot)$. and identify $\mathfrak{t}$ and $\mathfrak{t}^*$ using this metric.

**Definition 2.1.** We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by $x \in \mathfrak{t} \mapsto \varphi_x \in \mathfrak{t}^*$ by $\varphi_x(y) = (x, y)$ and $\varphi \in \mathfrak{t}^* \mapsto x_\varphi \in \mathfrak{t}$ by $(x_\varphi, y) = \varphi(y)$. It is easy to see that these identifications are inverses of each other.

We then transport the metric from $\mathfrak{t}$ to $\mathfrak{t}^*$. That is to say or all $\varphi, \psi \in \mathfrak{t}^*$, we define $(\varphi, \psi) = (x_\varphi, x_\psi)$. By definition, this makes the identification of $\mathfrak{t}$ with $\mathfrak{t}^*$ is an isometry.

With these choices, we see that $\alpha^\perp \subset \mathfrak{t}^*$ is identified with $\mathrm{Ker}(\alpha) = x_\alpha^\perp \subset \mathfrak{t}$.

**Lemma 2.2.** *The identification $\mathfrak{t} \cong \mathfrak{t}^*$ determined by a Wel-invariant metric on $\mathfrak{t}$ is a Weyl-invariant isomorphism*

*Proof.* Let $w \in W$ and $x \in \mathfrak{t}$. Then for all $y \in \mathfrak{t}$ we have

$$(w\varphi_x)(y) = \varphi_x(w^{-1}y) = (x, w^{-1}y) = (wx, y) = \varphi_{wx}(y).$$

Since this is true for all $y \in \mathfrak{t}$, $w(\varphi_x) = \varphi_{wx}$. □

## 2.2 Formulae for the actions of Reflections on $\mathfrak{t}$ and $\mathfrak{t}^*$

We are now in a position to give formulae for the actions of the Weyl group on $\mathfrak{t}$ and $\mathfrak{t}^*$. Since the Weyl group is generated by reflections associated with roots, we need only write formulae for the actions of these elements.

**Claim 2.3.** *Let $\alpha$ be a root. The action of the reflection $w_\alpha$ on $\mathfrak{t}^*$ is given by*

$$\varphi \mapsto \varphi - \frac{2(\alpha, \varphi)\alpha}{(\alpha, \alpha)}.$$

*The action of $w_\alpha$ on $\mathfrak{t}$ is given by*

$$X \mapsto X - \frac{2\alpha(X)x_\alpha}{(\alpha, \alpha)}.$$

*Proof.* The action of $w_\alpha$ on $\mathfrak{t}$ actis as an orthogonal reflection in $\ker(\alpha)$. Under the equivariant isometry $\mathfrak{t} \cong \mathfrak{t}^*$, the action of $w_\alpha$ on $\mathfrak{t}$ is sent to its action on $\mathfrak{t}^*$ and $\ker(\alpha)$ is sent to $\alpha^\perp$. Thus, the action of $w_\alpha$ on $\mathfrak{t}^*$ is an orthogonal reflection in $\alpha^\perp$. The first formula given above determines a linear map $\mathfrak{t}^* \to \mathfrak{t}^*$ that clearly fixes $\alpha^\perp$ point-wise and sends $\alpha \to -\alpha$. Since $\alpha$ is perpendicular to $\alpha^\perp$, the formula determines the orthogonal reflection in $\alpha^\perp$.

The action of $w_\alpha$ on $\mathfrak{t}$ is an orthogonal reflection in $\ker(\alpha)$. The second formula determines a linear map that clearly fixes $\ker(\alpha)$ and, since $\alpha(x_\alpha) = (\alpha, \alpha)$, sends $x_\alpha$ to $-x_\alpha$. Since $x_\alpha$ is orthogonal to $\ker(\alpha)$, the formula produces the orthogonal reflection in $\ker(\alpha)$. □

## 2.3 Two Fundamental Results

Now that we have explicit formulae for the action of the reflections on $\mathfrak{t}$ and $\mathfrak{t}^*$ we can derive the fundamental facts about all this structure.

**Claim 2.4.** *For every root $\alpha$ and every weight $\lambda \in \Lambda^*$, we have*

$$-\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

*Proof.* (Following Adams) Fix a root $\alpha$ and choose $v \in \mathfrak{t}$ a vector with $\alpha(v) = 2\pi$. Then $\exp(v) \in \ker(\alpha \colon T \to S^1)$. Hence, is fixed by $w_\alpha$ acting on $T$. This means that in $\mathfrak{t}$ we have

$$w_\alpha(v) - v = \frac{-2\alpha(v)x_\alpha}{(\alpha, \alpha)}$$

is an element of $\Lambda$. Hence, for any weight $\lambda \in \Lambda^*$ we have

$$\frac{-2\alpha(v)\lambda(x_\alpha)}{(\alpha, \alpha)} = \frac{-2\alpha(v)(\alpha, \lambda)}{(\alpha, \alpha)} \in 2\pi\mathbb{Z}.$$

Since $\alpha(v) = 2\pi$, we see that

$$-\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

□

**Claim 2.5.** *For roots $\alpha$ and $\beta$ we have*

$$\beta - \frac{2(\alpha, \beta)\alpha}{(\alpha, \alpha)}$$

*is a root, and*

$$\frac{-2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

*Proof.* The second statement follows from Claim 2.4. The first statement follows from Claim 2.3 and the fact that the Weyl group action preserves the roots. □

# 3 Root Systems

In this section we formalize the properties that we established in the previous section and the previous lecture.

## 3.1 Definitions

**Definition 3.1.** A *root system* consists of a finite dimensional real vector space $V$ and a finite set of non-zero vectors $R = \{\alpha_1, \dots, \alpha_k\}$ of $V$, *the roots*, a finite group $W$, *the Weyl group*, acting linearly and effectively on $V$ and a positive definite $W$-invariant inner product on $V$, satisfying the following axioms:

1. If $\alpha$ is a root then so is $-\alpha$ but no other real multiple of $\alpha$ is a root.
2. $W$ preserves the set of roots.
3. For each $\alpha \in R$ there is an element $w_\alpha \in W$ that is the reflection in the hyperplane orthogonal to $\alpha$.
4. The reflections associated with the roots generate $W$.
5. For each pair of roots $\alpha, \beta$ we have

$$-\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}.$$

We extend the basic terminology of roots, Weyl walls, Weyl chambers, and reflections to general root systems.

**Definition 3.2.** Given a root system $(V, R, W, (\cdot,\cdot))$, we say that $R$ is the *set of roots* and any element of $R$ is a *root*; $W$ is the *Weyl group*; for each root $\alpha \in R$ the linear subspace $W_\alpha = \alpha^\perp \subset V$ is the Weyl wall associated with $\alpha$ and the set of Weyl walls is the set of $W_\alpha$ for all roots $\alpha \in R$. Of course, $W_\alpha = W_{-\alpha}$. The Weyl chambers are the connected components of $V \setminus \cup_{\alpha \in R} W_\alpha$. These are open, convex subsets of $V$. For each $\alpha \in R$, the element $w_\alpha \in W$ is the *reflection* in $W_\alpha$. A Weyl wall $W_\alpha$ is a *wall of a Weyl chamber* $C$ if $\overline{C} \cap W_\alpha$ contains a point in no other Weyl wall beside $W_\alpha$. (Here, $\overline{C}$ is the closure of $C$.)

Clearly, if the root system is the root system associated with a compact, connected Lie group, these terms agree with the ones we are already using.

## 3.2 Comparison of Root Systems in General with those coming from compact, connected Lie groups

In one direction, the data coming from a connected, compact Lie group is a root system.

**Theorem 3.3.** *For any compact, connected Lie group $G$ with maximal torus $T$, the data* ($\mathfrak{t}^*$, roots of $T$, $W = W(G,T)$, $W$-invariant metric) *form a root system.*

*Proof.* The first four properties are established in Theorem 2.2 (Item 2), Proposition 4.7, Corollary 2.3, and Corollary 3.27, respectively, of the previous lecture. The last property is established in Claim 2.4 above. $\square$

In the other direction, things we established in the Lie group context hold for general root systems.

**Lemma 3.4.** *If $\alpha, \beta$ are roots and $\beta \neq \pm\alpha$, then the intersection $W_\alpha \cap W_\beta$ is a codimension-2 linear subspace of $V$ and is nowhere dense in $W_\alpha$.*

*Proof.* This is immediate from Axiom 1 for a root system. □

**Corollary 3.5.** *For any affine linear segment $\omega\colon [0,1] \to V$ with $\omega(i)$ in a Weyl chamber $C_i$ for $i = 0, 1$, there is an arbitrarily close generic affine linear segment. Theorem 3.14 holds for all root systems.*

*Proof.* Once we have Lemma 3.4, the proofs given for Lemma 3.13 and Theorem 3.14 in Lecture 8 go over *mutatis mutandis* □

**Lemma 3.6.** *In a general root system, for any root $\alpha$, the formula for reflection in Weyl wall $W_\alpha$ is given by*

$$v \mapsto v - \frac{2(\alpha, v)}{(\alpha, \alpha)}.$$

*In addition, Claim 2.5 of Lecture 8 holds for all pairs roots $\alpha, \beta$.*

*Proof.* It is a direct computation that the given formula is the formula for reflection in $\alpha^\perp$. Claim 2.5 from Lecture 8 then follows by applying this with $v = \beta$ and using the axiom that the roots are invariant under the Weyl action. □

# 4 Properties of Root Systems

We fix a root system $(V, R, W, (\cdot, \cdot))$.

**Lemma 4.1.** *Let $\alpha, \beta$ be a pair of roots with $\beta \neq \pm\alpha$. Then*

$$0 \leq \Big(\frac{-2(\alpha, \beta)}{(\alpha, \alpha)}\Big)\Big(\frac{-2(\beta, \alpha)}{(\beta, \beta)}\Big) < 4.$$

*Proof.* By Axiom 1, since $\beta \neq \pm\alpha$, $\beta$ and $\alpha$ are not real multiples of each other As an immediate consequence by Cauchy-Schwarz and the fact that the two terms have the same sign. $(\alpha, \beta)^2 < (\alpha, \alpha)(\beta, \beta)$. □

**Corollary 4.2.** *Let $\alpha, \beta$ be roots with $\beta \neq \pm\alpha$ and with $(\alpha, \beta) \leq 0$. Then either $(\alpha, \beta) = 0$ or one of*

$$\frac{-2(\alpha, \beta)}{(\alpha, \alpha)} \quad \text{or} \quad \frac{-2(\beta, \alpha)}{(\beta, \beta)}$$

*is equal to 1 and the other takes value in the set $\{1, 2, 3\}$.*

**Proposition 4.3.** *1. If $(\alpha, \beta) = 0$, then the angle between $\alpha$ and $\beta$ is $\pi/2$, the reflections in the hyperplanes perpendicular to $\alpha$ and $\beta$ commute and generate dihedral group of order 4.*
*2. Suppose that $\beta \neq \pm\alpha$ and*

$$\frac{-2(\beta, \alpha)}{(\beta, \beta)} = 1.$$

*Then the angle between $\alpha$ and $\beta$ is $\pi/3$, $\pi/4$, or $\pi/6$ depending on whether*

$$v = \frac{-2(\alpha, \beta)}{(\alpha, \alpha)} = 1, \quad 2, \quad \text{or } 3.$$

*Also, $\frac{|\beta|}{|\alpha|} = \sqrt{v}$. In these cases the reflections in the hyperplanes perpendicular to $\alpha$ and $\beta$ generate a dihedral group of order $6, 8$, or $12$.*
*3. Furthermore, under the hypothesis of previous statement, $\beta + k\alpha$ is a root for all*

$$0 \leq k \leq \frac{-2(\alpha, \beta)}{(\alpha, \alpha)}.$$

*Proof.* The first statement is clear.. Let us consider Statement 2. Let $\theta$ be the angle between $\alpha$ and $\beta$, so that $0 \leq \theta \leq \pi/2$. Set

$$v = \frac{-2(\alpha, \beta)}{(\alpha, \alpha))}.$$

Then

$$\cos^2(\theta) = \frac{v}{4},$$

or equivalently

$$\cos(\theta) = \pm\frac{\sqrt{v}}{2}.$$

Statement 2 is now clear.

If $\frac{-2(\alpha,\beta)}{(\alpha,\alpha)} = 1$, then the reflection of $\beta$ is the hyperplane perpendicular to $\alpha$ is $\beta + \alpha$.

If $\frac{-2((\alpha,\beta)}{(\alpha,\alpha)} = 2$, then reflection of $\alpha$ in the hyperplane perpendicular to $\beta$ is $\beta+\alpha$, whereas the reflection of $\beta$ in the hyperplane perpendicular to $\alpha$ is $\beta+2\alpha$.

Finally, if $\frac{-2(\alpha,\beta)}{(\alpha,\alpha)} = 3$, the reflection of $\alpha$ in the hyperplane perpendicular to $\beta$ is $\alpha+\beta$, and reflection of $\alpha+\beta$ in the hyperplane perpendicular to $\alpha$ is $\beta+2\alpha$. Lastly, reflection of $\beta$ in the hyperplane perpendicular to $\alpha$ is $\beta+3\alpha$.

Since the roots are invariant under the Weyl group action, this establishes Statement 3 in all cases. □

# 5 Positive Roots and Simple roots

To study root systems further, we fix a Weyl chamber and describe things in terms of that choice. Of course, all Weyl chambers are equivalent under the action of the Weyl group, so different choices lead to an isomorphic descriptions up to conjugation.

**Definition 5.1.** We fix a Weyl chamber $C_0$, called the *fundamental* Weyl chamber. Since $C_0$ is disjoint from the walls defined by the roots, each root is either positive or negative on $C_0$. Those that are positive on $C_0$ are called *positive* roots, and those that are negative are called *negative* roots (relative, of course, to $C_0$ which we consider as fixed for this discussion).

**Remark 5.2.** Every root is either positive or negative and the involution $-1$ on $V$ interchanges positive and negative roots. Thus, each wall is defined as the kernel of a unique positive root.

Since Theorem 3.14 of Lecture 8 holds for a general root system we have:

**Claim 5.3.** *Let $\alpha_1,\ldots,\alpha_k$ be the positive roots defining the walls of $C_0$. Then $C_0 = \cap_{i=1}^k\{\alpha_i > 0\}$.*

**Lemma 5.4.** *The fundamental Weyl chamber $C_0$ is the only chamber on which all positive roots are positive.*

*Proof.* Let $C$ be a chamber distinct from $C_0$. We claim that there is a wall separating $C_0$ and $C$. If not then all the positive roots defining walls of $C_0$ are positive on $C$ and hence $C \subset C_0$ which implies $C = C_0$. If the wall $W_\alpha$ associated to a positive root $\alpha$ separates $C$ and $C_0$, then $\alpha$ is negative on $C$. □

**Lemma 5.5.** *If $\alpha$ and $\beta$ are positive roots and $\alpha+\beta$ is a root, then $\alpha+\beta$ is a positive root. A non-trivial real linear combination of positive roots, $\sum_i \lambda_i\alpha_i$ with the $\lambda_i > 0$ is never the zero element of $V$.*

*Proof.* The first statement is obvious from the definition. As to the second, any non-trivial positive real linear combination of positive roots takes positive value at any point of $C_0$. □

**Definition 5.6.** A positive root is a *simple* root if it cannot be written as a sum of two positive roots.

The two main results we prove are:

- The simple roots are linearly independent and span the same subspace of $V$ as all the roots, which is the orthogonal complement of the subspace on which the Weyl group acts by the identity.
- The Weyl walls associated with the simple roots are exactly the walls of the fundamental chamber.

## 5.1 Span of the Simple Roots

**Lemma 5.7.** *Every positive root is a sum of simple roots.*

*Proof.* Suppose that $\alpha$ is a positive root that cannot be written as a sum of simple roots. Then $\alpha$ is not simple so that it can be written as a sum $\beta_1+\beta_1'$ of positive roots. If each of $\beta_1$ and $\beta_1'$ can be written as a sum of simple roots then so can $\alpha$. Thus, renumbering if necessary, we can assume $\beta_1$ cannot be written as a sum of simple roots, and, in particular is not a simple root.

Repeating the argument, we have $\beta_1=\beta_2+\beta_2'$ with $\beta_2$ and $\beta'$ positive roots and $\beta_2$ not a sum of simple roots. Assuming that $\alpha$ is not a sum of simple roots, inductively we creating a sequence $\{\beta_i=\beta_{i+1}+\beta_{i+1}'\}_{i=1}^{\infty}$ with each $\beta_i$ and $\beta_i'$ positive roots and $\beta_i$ not expressible as a sum of simple roots.

**Claim 5.8.** *In any expression for $\alpha$ as a positive sum of two or more positive roots the coefficient of $\alpha$ in the sum is $0$.*

*Proof.* Otherwise, $\alpha=\alpha+\mu$ where $\mu$ is a positive sum of positive roots.Then $\mu=0$ which by Lemma 5.5 means $\mu$ is the trivial sum. □

Since

$$\begin{aligned} \alpha &= \beta_1' + \cdots + \beta_{i-1}' + (\beta_i + \beta_i') \\ \beta_j &= \beta_{j+1}' + \cdots \beta_{i-1}' + (\beta_i + \beta_i') \quad \text{for} \quad j < i \end{aligned}$$

it follows from the previous claim $\alpha, \beta_1, \beta_2, \ldots$ are all distinct. Since there is a finite number of roots, this is a contradiction. □

**Lemma 5.9.** *If $\alpha$ and $\beta$ are simple roots, then $(\alpha, \beta) \leq 0$.*

*Proof.* If $(\alpha, \beta) > 0$, the by Part 3 of Proposition 4.3 the element $\beta - \alpha$ is a root. Either $\beta - \alpha$ or $\alpha - \beta$ is a positive root and consequently, either $\beta = \alpha + (\beta - \alpha)$ or $\alpha = \beta + (\alpha - \beta)$ is not simple. □

**Definition 5.10.** We denote by $S$ be the set of simple roots.

**Proposition 5.11.** *The simple roots are linearly independent.*

*Proof.* Suppose we have a linear relation $\sum_{\alpha \in S} \lambda_\alpha \alpha = 0$. We form two disjoint subsets of simple roots: $S_+ = \{\alpha | \lambda_\alpha > 0\}$ and $S_- = \{\alpha | \lambda_\alpha < 0\}$. Then define $v$ by

$$v = \sum_{\alpha \in S_+} \lambda_\alpha \alpha = \sum_{\alpha \in S_-} -\lambda_\alpha \alpha$$

with both sides having positive numerical coefficients. We have

$$(v, v) = \sum_{(\alpha, \beta) \in S_+ \times S_-} \lambda_\alpha \lambda_\beta (\alpha, \beta) \leq 0.$$

This implies that $v = 0$, and consequently that $\sum_{\alpha \in S_+} \lambda_\alpha \alpha = 0$. It follows from Lemma 5.5 that $S_+ = \emptyset$. The same argument shows that $S_- = \emptyset$, proving the linear independence of the simple roots. □

**Corollary 5.12.** *The simple roots are a basis for the orthogonal space to the subspace on which the Weyl group acts trivially, and*

$$\cap_{\alpha \in S} \{\alpha > 0\}$$

*is the fundamental Weyl chamber.*

*Proof.* Since every root is either a positive linear combination or negative linear combination of the simple roots, the simple roots span the same space as all the roots. This is clearly the orthogonal complement to the maximal linear subspace on which Weyl group action is trivial. By the linear independence of the simple roots, they form a basis for this subspace.

### 5.2 The Weyl walls of the simple roots are exactly the walls of the fundamental chamber

Let $\alpha_1, \cdots, \alpha_k$ be the simple roots. Then by Lemma 5.7 every positive root is positive on $D = \cap_{i=1}^{k}\{\alpha_i > 0\}$. Thus, every negative root is negative on $D$.. This means that no wall meets $D$ and hence, $D$ is contained in a Weyl chamber $C$. On the other hand, $\overline{D} \setminus D$ is contained in the union of the walls so that no connected open subset of $V$ that properly contains $D$ is contained in a chamber. It follows that $D$ is a chamber. Since the $\alpha_i$ are linearly independent each wall $\{\alpha_i = 0\}$ contains an open subset in the closure of $D$. Thus, each $\{\alpha_i = 0\}$ is a wall of $D$. □

## 6 The Dynkin diagram

**Definition 6.1.** The Dynkin diagram has nodes and connections between the nodes. The nodes are indexed by the simple roots. Two nodes have no connection if the roots that index the nodes are orthogonal. Two nodes, indexed by $\alpha, \beta$ have a single line connection between them if the angle between $\alpha$ and $\beta$ is $\pi/3$; they have a double line connection between them if the angle between $\alpha$ and $\beta$ is $\pi/4$, and they have a triple line connection between them if the angle between $\alpha$ and $\beta$ is $\pi/6$. In the last two cases we add an arrow to the multiple connection that points toward the shorter root.