Lie Groups: Fall, 2025
Lecture VIII:
Reflections, Walls, and Chambers for Compact, Connected Lie Groups

October 13, 2025

All of the structure of reflections, walls, and chambers comes from rank-one groups.

1 Rank-1 groups

Theorem 1.1. *Let G be a compact, connected Lie group of rank 1. Then G is isomorphic to one of the three following groups:*

- S^1
- $SO(3)$
- S^3.

In the second and third case there is one pair of roots for G and the Weyl group is a group of order 2 acting on the maximal torus by $t \mapsto t^{-1}$.

Proof. Let $T \subset G$ a maximal torus. Then G/T is even dimensional. We claim that this dimension is either 0 or 2. If the dimension is 0, then since G is connected $T = G$ and we have the first of the groups listed above. Suppose that the dimension of G is $n > 0$. Choose a positive definite inner product on $\mathfrak{g}$ that is invariant under the adjoint representation. Denote by $S(\mathfrak{g})$ be the unit sphere about 0 in $\mathfrak{g}$. It is diffeomorphic to S^{n-1}.

Let v be a unit vector in $\mathfrak{t}$. The map $\widetilde{\rho}\colon G \to S^{n-1}$ defined by $g \mapsto \mathrm{ad}(g)(v)$ factors to define a smooth map $\rho\colon G/T \to S(\mathfrak{g})$. We claim this map is one-to-one and is a local diffeomorphism. For if $\mathrm{ad}(g_1)(v) = \mathrm{ad}(g_2)(v)$, then $\mathrm{ad}(g_1^{-1}g_2)(v) = v$ and $\mathrm{ad}(g_1^{-1}g_2)$ fixes $\mathfrak{t}$ and hence $g_1^{-1}g_2$ commutes with T. By Corollary 2.4 of Lecture 7 the closure of abelian group generated by $g_1^{-1}g_2$ and T has a topological generator and hence is contained in a torus. Since T is a maximal torus, this implies that $g_1 g_2^{-1} \in T$. This proves that $\rho\colon G/T \to (\mathfrak{g})$ is one-to-one.

Since $\widetilde{\rho}(\exp(tX)(v) = \mathrm{Ad}(\exp(tX))(v)$, differentiating at the identity gives $d_e\widetilde{\rho}(X) = [X, v]$. Thus, the kernel of the differential $d_e\widetilde{\rho}$ at $e \in G$ is kernel of $[v, \cdot]$. The kernel of this map is the subspace fixed by the adjoint action of T, which, by Theorem 3.5 of Lecture 7, is $\mathfrak{t}$. This proves that the kernel of $d_e\widetilde{\rho}$ is $\mathfrak{t}$ and hence that the kernel of $d_{[e]}\rho$ is trivial. Since the dimensions of the domain and range are the same this implies that $d_{[e]}\rho$ is an isomorphism. Direct computation shows that $d_g(\widetilde{\rho})(g \cdot X) = \mathrm{Ad}(g)(d_e(\widetilde{\rho})(X)) = \mathrm{ad}(g)([X, v])$, so that the kernel of $d_g(\widetilde{\rho})$ is one-dimensional and hence $d_{[g]}(\rho)$ is injective for every $g \in G$. The same dimension count shows that $d_{[g]}(\rho)$ is an isomorphism for every $[g] \in G/T$.

Since both G/T and $S(\mathfrak{g})$ are compact, connected manifolds of dimension $(n-1)$ and $S(\mathfrak{g})$ is simply connected, it follows that ρ is a diffeomorphism. In particular, there is $w \in G$ with $\mathrm{ad}(w)(v) = -v$. Thus, $w \in N_G(T)$ and its image in $\overline{w} \in W(G, T)$ acts on T by sending θ to θ^{-1}.

Connecting w by a path $w(t)$ to e, we have a path of homomorphisms $\mathrm{Ad}(w(t))\colon S^1 \to G$ from the identity map to the inverse map. From this it follows that the map $\pi_1(T) \to \pi_1(G)$ sends twice the generator of $\pi_1(T)$ to the trivial element in $\pi_1(G)$. From the long exact sequence of a fibration

$$\pi_2(G/T) \to \pi_1(T) \to \pi_1(G)$$

we conclude that $\pi_2(G/T) \neq 0$. Since G/T is homeomorphic to S^{n-1} we conclude that $n = 3$.

The adjoint map is a homomorphism $G \to SO(\mathfrak{g}) \cong SO(3)$ with kernel equal to the center of G. Also, the center of G is finite since G has rank 1 and is not abelian. In particular, the adjoint form of G (by definition $G/(\mathrm{center}(G))$) is three-dimensional and is a subgroup of $SO(3)$. This shows that the adjoint form of G is $SO(3)$. Since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, it follows that $G \equiv SO(3)$ or S^3, the simply connected double cover of $SO(3)$.

The statement about the roots and Weyl group follow immediately. $\square$

2 Reflections in $W(G, T)$

Our next step is to extend the results about reflections for rank-one Lie groups to general compact, connected Lie groups. For this we begin with a technical lemma.

Lemma 2.1. *Let K be a compact, connected Lie group. Let $T \subset K$ be a maximal torus and let $H \subset T$ be a normal sub Lie group of K. Then:*

- *The pre-image in K of $N_{K/H}(T/H)$ is $N_K(T)$.*
- *T/H is a maximal torus of K/H.*
- *The map $K \to K/H$ induces an identification*

$$W(K,T) = W(K/H, T/H).$$

Proof. Clearly, if $w \in K$ normalizes T, then the image, $\overline{w}$, of w in K/H normalizes T/H. Conversely, suppose that $\overline{w} \in K/H$ normalizes T/H and let $w \in K$ be a lift of $\overline{w}$. Then for $t \in T$, $wtHw^{-1} = t'H$ for some $t' \in T$. Since H is normal in K, the element w normalizes H. This implies that $wtw^{-1} = t'h'$ for some $h' \in H$. Since $H \subset T$, this implies that $wtw^{-1} \in T$. Since this is true for every $t \in T$, we conclude that that $w \in N_K(T)$. This establishes Item 1.

T/H is a torus in K/H. Let $U/H \subset K/H$ be a maximal torus in K/H containing T/H. Then U/H commutes with T/H and hence normalizes T/H. From Item 1 it follows that $T \subset U \subset N_K(T)$. Since T is a maximal torus of K, $\dim(T) = \dim(N_K(T)$. It follows that $\dim(T) = \dim(U)$ and hence $\dim(T/H) = \dim(U/H)$. Since both these groups are tori, they are equal.

Item 3 is immediate from Items 1 and 2. □

Theorem 2.2. *Let k be the rank of G. Let $T \subset G$ be a maximal torus and let α be a root for G. Let $\hat{U}_\alpha = \ker(\alpha \colon T \to S^1)$ and let U_α be its component of the identity.*

1. *U_α is a codimension-1 torus in T and the component group of $\hat{U}_\alpha$ is a cyclic.*
2. *The component of the identity of the normalizer of U_α, $N_G(U_\alpha)_0$, has dimension $k+2$ and there are the only roots of G that vanish on U_α, namely $\pm\alpha$.*
3. *T is a maximal torus of $N_G(U_\alpha)_0$ and $W(N_G(U_\alpha)_0, T) \cong \mathbb{Z}/2\mathbb{Z}$. Let w_α be the non-trivial element of $W(N_G(U_\alpha)_0, T)$. The adjoint action of w_α on T fixes U_α and acts by inverse on the quotient T/U_α.*

4. *$\hat{U}_\alpha$ centralizes $N_G(U_\alpha)_0$, and $N_G(\hat{U}_\alpha)_0 = N_G(U_\alpha)_0$.*

5. *The element w_α centralizes $\hat{U}_\alpha$. Also, $\hat{U}_\alpha$ has at most 2 components, and w_α acts on $T/\hat{U}_\alpha$ by inverse.*

Proof. 1). U_α is the component of the identity of $\ker(\alpha)$ and thus is a subtorus of T of codimension-1. The group of components of $\hat{U}_\alpha$ is cyclic group of order equal to the order of divisibilty of α in the group of characters.

2). Since the automorphism group of U_α is finite $N_G(U_\alpha)_0$ centralizes of U_α. Being a torus, U_α has a generator g. We have just seen that $N_G(U_\alpha)_0$ is contained in the $Z(g)$, the centralizer of g in G. Obviously, then it is contained in the component of the identity of $Z(g)$. The centralizer of g in G is the centralizer of U_α in G and is contained in $N_G(U_\alpha)$. Thus the component of the identity of $Z(g)$ is contained in $N_G(U_\alpha)_0$. This proves that $N_G(U_\alpha)_0$ is equal to the component of the identity of $Z(g)$.

According to Lemma 3.4 of Lecture 7, the Lie algebra of $Z(g)$ is equal to $\ker(\mathrm{ad}(g))$. This kernel is $\mathfrak{t} \oplus_{\{\beta|\beta(g)=1\}} V_\beta$. In particular, the dimension of $N_G(U_\alpha)_0$ is $k + \#\{\beta|\beta(g) = 1\}$. We must show that $\pm\alpha$ are the only roots sending g to the identity, or equivalently the only roots vanishing on U_α. Suppose that there were a second pair $\{\pm\beta\} \neq \{\pm\alpha\}$ vanishing on g. Then $\mathrm{ad}(g)$ vanishes on $\mathfrak{t} \oplus V_\alpha \oplus V_\beta$ the dimension of $Z(g) = N_G(U_\alpha)_0$ is at least $k + 4$. It follows from Lemma 2.1 that T/U_α is a maximal torus of $N_G(U_\alpha)_0/U_\alpha$. Thus, $N_G(U_\alpha)_0/U_\alpha$ is of rank 1 and dimension ≥ 5, contradicting Theorem 1.1. This shows that is exactly one pair of roots that vanish on U_α; namely $\pm\alpha$.

3). Since T is a maximal torus of G contained in $N_G(U_\alpha)_0$, it is a maximal torus of $N_G(U_\alpha)_0$. Lemma 2.1 identifies the Weyl group $W(N_G(U_\alpha)_0, T)$ with the Weyl group $W(N_G(U_\alpha)_0/U_\alpha, T/U_\alpha)$. But $N_G(U_\alpha)_0/U_\alpha$ is a group of rank one and dimension 3, and hence is isomorphic to $SO(3)$ or S^3, and the non-trivial element in the Weyl group acts by inverse on the maximal torus.

4). Under the adjoint representation $\hat{U}_\alpha$ acts trivially on the root space V_α. Since $\hat{U}_\alpha \subset T$, its adjoint action on $\mathfrak{t}$ is trivial. Being an extension of finite cyclic group by a torus, $\hat{U}_\alpha$ has a topological generator, say $v \in T$. The adjoint action of v on the Lie algebra $\mathfrak{t} \oplus V_\alpha$ of $N_G(U_\alpha)_0$ is trivial. It follows that v centralizes $N_G(U_\alpha)_0$ as does its closure $\hat{U}_\alpha$. Equivalently, $N_G(U_\alpha)_0$ centralizers $\hat{U}_\alpha$ and hence normalizes it. That is to say $N_G(U_\alpha)_0 \subset N_G(\hat{U}_\alpha)$. It follows that $N_G(U_\alpha)_0 \subset N_G(\hat{U}_\alpha)_0$.

Conversely, any element that normalizes $\hat{U}_\alpha$ clearly also normalizes its connected component of the identity, namely, U_α. It follows that $N_G(\hat{U}_\alpha) \subset N_G(U_\alpha)$, and hence that $N_G(\hat{U}_\alpha)_0 \subset N_G(U_\alpha)_0$. This proves 4.

5). Since $w_\alpha \in N_G(U_\alpha)_0$, by Item 4 it centralizes $\hat{U}_\alpha$. Thus, $\mathrm{Ad}(w_\alpha)$ acts trivially on $\hat{U}_\alpha/U_\alpha \subset T/U_\alpha$. Since the adjoint action of w_α on T/U_α sends every element to its inverse, there are only two fixed points of the action. Thus, $\hat{U}_\alpha/U_\alpha$ has cardinality 1 or 2. Since w_α acts by inverse on T/U_α, it also acts by inverse on $T/\hat{U}_\alpha$. □

Corollary 2.3. *For each pair of roots $\pm\alpha$ of T, there is an element of order two $w_\alpha \in W(G,T)$ that fixes the kernel of $\alpha\colon T \to S^1$ pointwise and acts by inverse on the quotient $T/\ker(\alpha)$.*

Proof. Theorem 2.2 constructs exactly such an element in $W(N_G(U_\alpha)_0, T)$. Of course, $W(N_G(U_\alpha)_0, T)$ is naturally a subgroup of $W(G,T)$. □

Remark 2.4. Examining the proof above closely, one can see that we have given a complete description of $N_G(U_\alpha)_0$. We have the inclusion $i\colon U_\alpha \subset N_G(U_\alpha)_0$ whose image is a central subgroup. We also have the 3-dimensional Lie algebra generated by a basis X, Y of V_α. The element $Z = [X,Y] \in \mathfrak{t}$ and the line it generates is complementary to $L(U_\alpha)$, the Lie algebra of U_α. Since the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$ stabilizes V_α, we see that $\mathbb{R}(Z) \oplus V_\alpha$ is closed under bracket and is the Lie algebra $\mathfrak{so}(3)$. Hence, there is a map of Lie groups $\rho\colon S^3 \to N_G(U_\alpha)_0$ whose Lie algebra image is exactly this $\mathfrak{so}(3)$. Since $i\colon U_\alpha \subset N_G(U_\alpha)_0$ is central, we can form the product map

$$i \times \rho\colon U_\alpha \times S^3 \to N_G(U_\alpha)_0.$$

This Lie group homomorphism induces an isomorphism on Lie algebras. The kernel of the homomorphism is a discrete central subgroup A. Since $i\colon U_\alpha \subset N_G(U_\alpha)_0$ is an injection, $A \cap U_\alpha = \{e\}$. This means that the projection onto S^3 induces an injection from $A \to S^3$ whose image obviously lies in the center of S^3. There are two possibilities for $N_G(U_\alpha)_0$:

- $A = \{e\}$ and $N_G(U_\alpha)_0 \cong U_\alpha \times S^3$.
- $A \cong \mathbb{Z}/2\mathbb{Z}$ and $N_G(U_\alpha) \cong U_\alpha \times_A S^3$, where the projection to S^3 induces an isomorphism to A to the center of S^3 and the projection of A to U_α is an element of order 2 or 1 of T.

In the first case $\hat{U}_\alpha$ has two components and in the second it has one.

Definition 2.5. The *derived sub-algebra* of a Lie algebra L is the sub-algebra $[L,L]$, the $\mathbb{R}$-linear span over of all brackets of two elements in L.

Corollary 2.6. *The derived sub-algebra of the Lie algebra of $N_G(U_\alpha)_0$ is $\mathfrak{so}(3)$ generated by a basis X, Y of $V_\alpha \subset \mathfrak{g}$. Furthermore, the line spanned by $[X, Y]$ is contained in $\mathfrak{t}$, is invariant under w_α, and the action of w_α on this line is multiplication by -1.*

Proof. Since the Lie algebra of $N_G(U_\alpha)_0$ is isomorphic to $L(U_\alpha) \oplus \mathfrak{so}(3)$, it is clear that its derived algebra is $\mathfrak{so}(3)$. A direct computation with the presentation of $\mathfrak{so}(3)$ shows that under the decomposition $\mathfrak{so}(3) = \langle [X, Y]\rangle \oplus V_\alpha$, the bracket $[X, Y]$ lies in $\mathfrak{t}$. The inclusion of $\mathfrak{so}(3)$ into the Lie algebra of $N_G(U_\alpha)_0$ integrates to give a map $S^3 \to N_G(U_\alpha, T)_0$ that sends the Weyl group of S^3 isomorphically onto the Weyl group of $W(N_0(_\alpha), T)$. Thus, the line spanned by $[X, Y]$ in $\mathfrak{t}$ is invariant under the Weyl group and the non-trivial element of this groups acts by -1 on the line spanned by $[X, Y]$. $\square$

Corollary 2.7. *Let $T \subset G$ be a maximal torus. Any element in $g \in T$ that is not in the kernel of any root is contained in no maximal torus distinct from T. If $g \in \ker(\alpha) \subset T$ for some root α, then g is contained in at least two distinct maximal tori.*

Proof. If $g \in T$ is not contained in the kernel of any root, then g acts non-trivially on each root space V_α. Thus, the subspace of $\mathfrak{g}$ on which $\mathrm{ad}(g)$ acts by the identity is $\mathfrak{t}$. By Lemma 3.4 of Lecture 7, this means that the Lie algebra of the centralizer $Z(g)$ is $\mathfrak{t}$. This means that the component of the identity of $Z(g)$ is T. Obviously, then g is contained in only one maximal torus, which is T.

Now suppose that $g \in \ker(\alpha)$. Then $g \in \hat{U}_\alpha$ which, by Item 4 of Theorem 2.2, implies that g centralizes $N_G(\hat{U}_\alpha)_0$. Since $g \in T \subset N_G(T)_0$, g is in the center of $N_G(U\alpha)_0$. Thus, g is contained in every maximal torus $N_G(U_\alpha)_0$. All maximal tori of $N_G(U_\alpha)_0$ are conjugate in $N_G(U_\alpha)_0$ and hence are conjugate in G. Since T is a maximal torus of $N_G(U_\alpha)_0$, all the maximal tori of $N_G(U_\alpha)_0$ are maximal tori of G. Since $\dim(N_G(U_\alpha)_0 > \dim(T)$, there is more than one maximal torus of $N_G(U_\alpha))$. (In fact, there is a two-dimensional family of them.) $\square$

3 Weyl Chambers, Weyl Walls and Reflections

Now we pass from a study of the roots and Weyl group action on the maximal torus to a study of the analogous objects in $\mathfrak{t}$, the Lie algebta of T.

Remark 3.1. In Lecture 7 and so far in this lecture, the roots refer to non-trivial characters $\alpha_j \colon T \to S^1$ that occur in the decomposition of the adjoint action of T on its Lie algebra $\mathfrak{t}$. In this section we shall discuss the lifts of these to homomorphisms $\mathfrak{t} \to \mathbb{R}$ of the universal covers, where they have the condition that restricted to $\Lambda \subset \mathfrak{t}$ they lie in $\mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$. In fact, lifting in this fashion gives an isomorphism $\Lambda^* = \mathrm{Hom}(T, \S^1) = \mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$

By abuse of notation we shall use the term *roots* to also mean the maps $\mathfrak{t} \to \mathbb{R}$ induced by roots $\alpha_j \colon T \to S^1$ and denote them by the same symbols. It should be clear from context when we introduce a 'root' α_j in which context we are working.

3.1 The Definitions

We fix a compact, connected Lie group G and a maximal torus $T \subset G$. Since $W(G,T)$ is a finite group, there is a positive definite $W(G,T)$-invariant quadratic form on $\mathfrak{t}$. We fix one and denote the associated non-degenerate bilinear form by $(\cdot, \cdot)$.

Definition 3.2. Let $R \subset \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ be the set of roots for (G,T). For each root $\alpha \in R$, let W_α be the kernel of $\alpha \colon \mathfrak{t} \to \mathbb{R}$. These are the *Weyl walls*. Notice $W_\alpha = W_{-\alpha}$.

One relationship between the two notions of roots is obvious. It is:

Lemma 3.3. *For each root α, the exponential map induces a covering $W_\alpha \to U_\alpha$, where, as above, U_α is the component of the identity of* $\ker(\alpha \colon T \to S^1)$.

Lemma 3.4. *There are only finitely many Weyl walls and each is a codimension-1 linear subspace of $\mathfrak{t}$. If β and α are roots and $\beta \neq \pm\alpha$, then $W_\alpha \neq W_\beta$ For each Weyl wall W_α, there is an element $w_\alpha \in W(G,T)$ that acts as a reflection in W_α with respect to the $W(G,T)$-invariant bilinear form*

Proof. Since there are only finitely many roots α, there are finitely many Weyl walls W_α. The reflection w_α is produced by Corollary 2.3. It follows from Part 2 of Theorem 2.2 that if $\beta \neq \pm\alpha$ then β is non-trivial restricted to $\ker(\alpha) \subset T$. By Lemma 3.3, this implies the same result when we lift to $\mathfrak{t}$. □

Definition 3.5. A *Weyl chamber* is a connected component of $\mathfrak{t} \backslash (\cup_{\alpha \in R} W_\alpha)$.

Lemma 3.6.
1. *For any Weyl chamber $C \subset \mathfrak{t}$ and any root α, either $\alpha > 0$ on C or $\alpha < 0$ on C.*
2. *If x and y are points not lying on any Weyl wall, then x and y lie in the same Weyl chamber if and only if the sign of $\alpha(x)$ equals that of $\alpha(y)$ for every root α.*

3. *The Weyl chambers are open, convex subsets of* $\mathfrak{t}$.

Proof. 1. Since every Weyl chamber C is connected, if there are $x, y \in C$ and a root α with $\alpha(x) < 0 < \alpha(y)$, then by the intermediate value them there is a point $z \in C$ with $\alpha(z) = 0$. But this is absurd since that means $z \in W_\alpha$ whereas z is in a Weyl chamber and hence is disjoint from all Weyl walls.

2. In Item 1 we saw that each α is either positive or negative on all of any Weyl chamber C. Suppose now that x, y are points not in any Weyl wall such that for all α the sign of $\alpha(x)$ agrees with that of $\alpha(y)$. Let γ be the affine linear line path from x to y. Then $\alpha|_\gamma$ is an affine linear function with the same sign at the endpoints. Hence, it has a constant sign along γ. Thus, x and y are in the same connected component of the complement of the Weyl walls; i.e., in the same Weyl chamber.

3. Being the finite intersections of open half spaces in $\mathfrak{t}$, the Weyl chambers are open, convex subsets of $\mathfrak{t}$. $\square$

Remark 3.7. For a Weyl wall W, the complement, $\mathfrak{t} \setminus W$, is the disjoint union of two open half-spaces. If there are k Weyl walls we define 2^k different subsets of $\mathfrak{t}$ as follows. The subsets are indexed by a choice for each Weyl Wall of one of its two complementary open half-spaces. The subset associated to an index is the intersection of the chosen open half-spaces. For many of the indices, the corresponding subset of $\mathfrak{t}$ is empty. The non-empty intersections are exactly the Weyl chambers, and each Weyl chamber occurs exactly once in the indeed collection.

3.2 The Weyl Group Action on the set of Weyl Chambers

Definition 3.8. A Weyl W_α is a *wall of a Weyl chamber* C if and only if intersection of the closure of C and W_α contains a point in no Weyl wall distinct from W_β for every root $\beta \neq \pm\alpha$..

Lemma 3.9. *Fix a Weyl wall* W_α. *Then for any other Weyl wall* W_β *with* $\beta \neq \pm\alpha$*, the intersection* $W_\beta \cap W_\alpha$ *is a codimension-*1 *linear subspace of* W_α. *The union of these intersections as* W_β *ranges over Weyl walls distinct from* W_α *is a closed nowhere dense subset of* W_α.

Proof. The first statement is obvious. The second follows from it and the fact that there are only finitely many Weyl walls. $\square$

Definition 3.10. For each root α we define $V_\alpha \subset W_\alpha$ to be the complement of the intersection of W_α with the union of Weyl walls distinct from W_α.

Lemma 3.11. *If W_α is a wall of a Weyl chamber C, then the closure of C meets W_α in closure the of one of the components of V_α. The reflection w_α takes C to a Weyl chamber C' that also has W_α as a wall.*

Proof. Let $x \in W_\alpha \cap \overline{C}$ be a point in no other Weyl wall. Then there is a convex open neighborhood U of x in $\mathfrak{t}$ that meets no Weyl wall other than W_α. For each Weyl wall W_β distinct from W_α there is a unique open half-space of the complement of W_β that contains U. The intersection, A, of these open half spaces as we range over roots β distinct from $\pm\alpha$ meets W_α in exactly in one of the components of V_α, the component containing U.

Consider the intersections $A \cap \{\alpha > 0\}$ and $A \cap \{\alpha < 0\}$. Each of these is non-empty since it contains a non-empty subset of U. Thus, each of these two intersections is a Weyl chamber, and these are the only two Weyl chambers whose closure contains $x \in W_\alpha$. One of them is the original Weyl chamber C and the other is its image under the reflection in W_α. Each of these has W_α as a wall and in fact the closure of each meets W_α in the closure $A \cap W_\alpha$. □

Definition 3.12. We say that an affine linear segment ω in $\mathfrak{g}$ is generic if:

- each end point of ω is contained in Weyl chamber;
- ω does not meet the intersection of any two distinct Weyl walls.

Lemma 3.13. *Let $\omega\colon [0,1] \to \mathfrak{t}$ be an affine linear segment with end points in Weyl chambers $\omega(0) \in C$ and $\omega(1) \in C'$. Then there is an arbitrarily close, generic affine linear segment. It has end points in the same Weyl chambers as ω does.*

Proof. Fixing one endpoint, the condition that an affine linear segment meet a codimension-2 linear subspace is a single affine linear condition on the other endpoint of ω. The result follows easily. □

Theorem 3.14. *Let C be a Weyl chamber and let $W_{\alpha_1} \ldots W_{\alpha_k}$ be the walls of C, with the α_i chosen so that $\alpha_i|_C > 0$. Then*

$$C = \cap_{i=1}^{k} \{\alpha_i > 0\}.$$

Proof. We begin the proof of the theorem with a lemma.

Lemma 3.15. *Let ω be a generic affine linear segment with $\omega(0) \in C$. If ω crosses a Weyl wall, then the first Weyl wall it crosses is a wall of C.*

Proof. Suppose that ω crosses a Weyl wall and let $t \in (0,1]$ be the smallest number with the property that $\omega(t)$ is in a Weyl wall, say W_β with $\beta > 0$ on C. Clearly, $\omega(s) \in C$ for $s \in [0,t)$ and $\omega(t) \notin C$, so that $\omega(t) \in \overline{C} \setminus C$. . Since ω is generic, $\omega(t)$ does not lie in any $W_\beta \cap W_\gamma$ for any root $\gamma \neq \pm\beta$. This proves that W_β is a wall of C. □

Turning to the proof of the theorem, since the α_i are positive on C, it is clear that $C \subset \cap_{i=1}^k \{\alpha_i > 0\}$. We must prove the converse. Arguing by contradiction, suppose that C is properly contained in $\cap_{i=1}^k \{\alpha_i > 0\}$. Then some Weyl wall W_β must meet $\cap_{i=1}^k \{\alpha_i > 0\}$. For if not, then $\cap_{i=1}^k \{\alpha_i > 0\}$ is contained in a Weyl chamber which, since it contains C, must then be C.

We choose β so that it is positive on C. Let $q \in W_\beta \cap_{i=1}^k \{\alpha_i > 0\}$. Then there is a generic affine linear line segment with one endpoint $p \in C$ and the other endpoint arbitrarily close to q with $\beta(q) < 0$. This segment is contained in $\cap_{i=1}^k \{\alpha_i > 0\}$. As such it crosses no wall of C. But it crosses W_β. This contradicts Lemma 3.15. □

Theorem 3.16. *The action of the Weyl group on* $\mathfrak{t}$ *sends Weyl chambers to Weyl chambers. The Weyl group acts simply transitively on the set of Weyl chambers.*

Proof. According to Proposition 4.7 of Lecture 7, the action of the Weyl group preserves the set of roots and hence stabilizes the $\cup_{\alpha \in R} W_\alpha$. Consequently, it preserves the union of the Weyl chambers and hence permutes the Weyl chambers, which are its connected components.

Suppose that C is a Weyl chamber fixed by $w \in W(G,T)$ with $w \neq e$. Let $n > 1$ be the order of w. Let $v \in C$ and consider the average

$$\hat{v} = \frac{1}{n} \sum_{k=1}^{n} w^k v.$$

Since $w^k v \in C$ for all k and C is convex, $\hat{v} \in C$ and is invariant under w.

Let H be the one parameter subgroup $\exp(t\hat{v})$. Since $\hat{v} \in C$, the open half-line $\mathbb{R}^+ v$ is disjoint for the Weyl walls. There is a neighborhood V of $0 \in \mathfrak{g}$ such that the exponential map $U \to G$ is a diffeomorphism onto an open set $V$and the intersection of U with kernel of a root $\alpha \colon \mathfrak{t} \to \mathbb{R}$ maps diffeomorphically onto the intersection of V with the kernel of $\alpha \colon T \to S$. Thus, $\exp(t\hat{v}) \in T$ is not in the kernel of any root $\alpha \colon T \to S^1$ for all $t \in (0, \epsilon)$.

Let $\widetilde{w} \in N_G(T)$ be a lift of w. Since $w \cdot \hat{v} = \hat{v}$, the element $\widetilde{w}$ commutes with H. This means that the group K generated by H and $\widetilde{w}$ is ablelian. It follows from Corollary 2.5 of Lecture 7 that K is contained in a maximal torus T'. Since $w \in W(G,T)$ and $w \neq e$, if follows that $\widetilde{w} \notin T$, and, as a result, $T \neq T'$. Thus, H is contained in two distinct maximal tori. That implies that $H \subset \cup_\alpha \hat{U}_\alpha$. This contradicts the fact that there is $\epsilon > 0$ such that for all $t \in (0,\epsilon)$ $\exp(tv)$ is not in the kernel of any root. This is a contradiction, showing that no non-trivial element of $W(G,T)$ fixes any Weyl chamber.

To complete the proof, we need to show that the Weyl group acts transitively on the set of Weyl chambers. For each Weyl wall W_α we have a reflection $w_\alpha \in W(G,T)$ and this reflection interchanges the two Weyl chambers that have W_α as a wall. Let C and C' be Weyl chambers. Chose a generic affine linear line segment $\omega\colon [0,1] \to \mathfrak{t}$ from a point of C to a point of C'. Let $C = C_0, C_1, \ldots C_n = C'$ be the Weyl chambers, in order, that this segment meets. At the point where it crosses from C_i to C_{i+1} it lies in some Weyl wall W_{α_i}. But since the segment is generic, it lies in no other Weyl wall. That is to say W_{α_i} is a wall of C_i and C_{i+1} and w_{α_i} maps C_i to C_{i+1}. Hence the product $w_{\alpha_{n-1}} w_{\alpha_{n-2}} \cdots w_{\alpha_0}$ maps $C = C_0$ to $C' = C_n$. $\square$

3.3 Weyl Group is Generated by Reflections in Weyl Walls

Corollary 3.17. *The reflections $\{w_\alpha\}$ in the roots $\alpha \in R$ generate the Weyl group.*

Proof. In the proof that the action of $W(G,T)$ on the set of Weyl chambers is transitive, we actually showed that the subgroup of the Weyl group generated by reflections in Weyl walls acts transitively. Since the action of the Weyl group is effective, it follows that the subgroup generated by reflections is the entire Weyl group. $\square$