Lie Groups: Fall, 2025
Lecture VII:
Compact Lie Groups, Maximal Tori, and Weyl Groups

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We begin with a basic lemma.

Lemma 0.1. *The component group of any compact Lie group is finite.*

Proof. Since a Lie group is a manifold and hence locally connected, each connected component of G is an open subset of G. Were there infinitely many connected components, this would give an infinite covering by disjoint open sets, contradicting compactness. □

1 Linear Actions of S^1 and Tori

1.1 Complex Actions of S^1

Identify the Lie algebra of S^1 with $\mathbb{R}$ and the exponential map with the usual map $\mathbb{R} \to S^1$ given by $t \mapsto \exp(it)$. Let $S^1 \times V \to V$ be a finite dimensional, complex linear action. The induced map on Lie algebras sends $1 \in \mathbb{R}$ to some $A \in M(n \times n, \mathbb{C})$. According to the Jordan canonical form, we can find a basis of V in which $A = A_{ss} + A_{nil}$ with A_{ss} is diagonalizable and A_{nil} is a strictly upper trianglar matrix commuting with A_{ss}. Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of A_{ss}. Since $\exp(2\pi A) = 1$, we see that each λ_j is of the form in_j for some integers n_j.

Since $\exp(itA)$ and $\exp(itA_{ss})$ are periodic of period 2π and since A_{ss} and A_{nil} commute, it follows that $\exp(itA) = \exp(itA_{ss})\exp(itA_{nil})$, so that (itA_{nil}) is also periodic of period 2π. On the other hand, since A_{nil} is strictly upper triangular, some power of A_{nil} is identically zero. Thus, $\exp(itA_{nil})$

a finite polynomial expression in itA_{nil} whose constant term is Id and the linear term is itA_{nil}. The only periodic polynomials of t are constant polynomials.. This implies that $A_{nil} = 0$ and $A = A_{ss}$ is diagonalizable with eigenvalues in_j for integers n_j.

Definition 1.1. A *character* of S^1 is a homomorphism $S^1 \to S^1$. The group of characters of S^1 is naturally identified with $\mathbb{Z}$ given by $\theta \mapsto \theta^n$.

A representation of S^1 on a complex vector space of dimension n is, up to conjugation, given by n characters. That is to say there is a basis in which the action of given by diagonal matrices, and each diagonal entry is a character of S^1.

1.2 Complex Actions of a Torus

Definition 1.2. By a *torus* we mean a compact, connected, abelian Lie group T. The Lie algebra $\mathfrak{t}$ of T is an abelian Lie algebra and hence the BCH series is $H(X,Y) = X+Y$. This converges on all of $\mathfrak{t}\times\mathfrak{t}$ and defines a group structure of $\mathfrak{t}$ which is the usual addition. The exponential map is a Lie group map from $\mathfrak{t}$ with its addition to T and is a local diffeomorphism. Hence, the kernel of exp is a discrete subgroup Λ of $\mathfrak{t}$ and the exponential mapping induces an isomorphism from $\mathfrak{t}/\Lambda \to T$. Since T is compact, $\Lambda \subset \mathfrak{t}$ must be a lattice; i.e., a discrete subgroup generated by an $\mathbb{R}$-basis $\{a_1, \ldots, a_n\}$ of $\mathfrak{t}$, where $n = \dim(\mathfrak{t})$.

Remark 1.3. The circle is a one-dimensional torus. Any torus is isomorphic as a Lie group to a finite product of circles with the product Lie group structure. (This is a homework problem.)

The results about complex actions of S^1 generalize to any torus T.

Definition 1.4. A *character of a torus* T is a homomorphism $T \to S^1$. If we write $T = \mathfrak{t}/\Lambda$ then a character of T is a linear map $\mathfrak{t} \to \mathbb{R}$ that sends $\Lambda \to 2\pi\mathbb{Z}$. The group of characters is the dual group $\Lambda^* = \mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$, to Λ. The formula for the character $T \to S^1$ associated to the linear function $\alpha\colon \mathfrak{t} \to \mathbb{R}$ sending $\Lambda \to \mathbb{Z}$ is

$$\exp(v) \mapsto \exp(i\alpha(v)).$$

Let $T \times V \to V$ be a complex linear action. We write the torus as a product of commuting circles. Let $X_1, \ldots, X_k$ be the elements of $\mathfrak{t}$ generating these circles as before. We have seen that each X_i is diagonalizable. Since the X_i commute, they have common eigenspaces. This means that we can find a basis for V, $\{e_1, \ldots, e_n\}$ so that T stabilizes each of the complex lines $\mathbb{C}e_i$. The action of T on $\mathbb{C}e_i$ is a by a character of T, and up to conjugation, an action of T on an n-dimensional vector space is the same as n characters of T.

Definition 1.5. The characters of this action are the *weights* of the representation of T on V.

1.3 Real Actions

Now let V be a finite dimension real vector space $S^1 \times V \to V$ be a real linear action. We can complexify the action and diagonalize the result:

$$V \otimes_{\mathbb{R}} \mathbb{C} = E_0 \oplus_{j \in I} E_{n_j}$$

where the action on E_0 is trivial and the action of the E_{n_j} are given by $e^{it} \cdot w = e^{in_j t} w$ for $w \in E_{n_j}$ a non-zero integer.. Since the action is real, we have $\overline{E}_{n_j} = E_{-n_j}$. In particular, E_0 is real, meaning that $E_0 = (E_0 \cap \mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, and each $E_{n_j} \oplus E_{-n_j}$ is real. The action of S^1 on $(E_0 \cap \mathbb{R}^n)$ is trivial. The intersection of $E_{n_j} \oplus E_{-n_j}$ with the real subspace projects equivariantly and isomorphically onto each of E_{n_j} and E_{-n_j}. Depending on the choice of which subspace we project onto, we see that the action of S^1 on this real subspace is given by e^{it} rotates by either e^{int} or e^{-int}. (These two actions are equivalent by the isomorphism $e^{it} \mapsto e^{-it}$ of S^1.)

This generalizes to tori. Any real linear action of a torus T on V is a direct sum of a trivial action and actions on two-dimensional spaces given by a character (i.e., a homomorphism) $\alpha_j \colon T \to S^1$ followed by the standard action of S^1 on $\mathbb{R}^2$. As in the case of the circle the character is only defined up to inverse. The weights of the real action are defined to be $\{\alpha_j^{\pm 1}\}_j$. In fact, these are the weights of the complexification of the representation.)

From now on we view characters of the torus $T = \mathfrak{t}/\Lambda$ **as** $\Lambda^* = \mathrm{Hom}(\Lambda, 2\pi\mathbb{Z})$ **and write characters additively instead of multiplicatively.**

2 Closures of Cyclic Subgroups

Definition 2.1. Let A be a Lie group. An element $g \in A$ is said to *generate A topologically* if the cyclic group generated by g is dense in A, or equivalently the closure of $\{g^n\}_{n \in \mathbb{Z}}$ is A. In this case we say that A has a *topological generator*.

Lemma 2.2. *If a subgroup A of a Lie group has a topological generator g, then A is abelian and its group of components is a cyclic group.*

Proof. Since all powers of g commute with each other, any element in A, the closure of the group generated by g, commutes with every power of g. Hence, A commutes with the closure of the group generated by g. This proves that eery element of A commutes with A, so that A is abelian.

Since the powers of g are dense in A, every component of A contains a power of g. Thus, the cyclic group generated by g maps onto the component group of A. Thus, the component group of A is also cyclic. $\square$

Corollary 2.3. *Every torus has a topological generator.*

Proof. Let T be a torus written as V/Λ, a vector space modulo a lattice Λ. A codimension-1 subtorus is determined by a linear map $\pi \colon V \to \mathbb{R}$ that induces a surjection $\pi_\Lambda \colon \Lambda \to 2\pi\mathbb{Z}$. The subtorus is the quotient of the kernel of π modulo the lattice by a lattice $\ker(\pi_\Lambda)$. There are only countably many such maps and subtori.

Consider the union over the countable collection of all such maps, π_Λ of $\pi^{-1}(2\pi\mathbb{Q}) \subset V$. This is a nowhere dense subset $\widetilde{D}$ invariant under the action of Λ. Let D be the image in T of $\widetilde{D}$. It is nowhere dense in T. For any $g \notin D$, no positive power of g is contained in a codimension-1 subtorus. Let C be the closure of $\{g^n\}_{n=1}^{\infty}$. This is an abelian sub Lie group of G. The component group of C is finite and hence some positive power of g is contained in the component of the identity C_0 of C. Being a connected, abelian Lie group C_0, is a torus. Since it contains a positive power of g, it follows from the fact that g and all its positive powers are in the complement of D that no positive power of g is contained in a proper subtorus of T. Thus, $C_0 = T$. $\square$

Corollary 2.4. *1. Let $A \subset G$ be an abelian Lie subgroup containing a torus T with finite cyclic quotient. Then A has a topological generator.*

2. If $A \subset G$ is the closure of an abelian group that is generated by a connected subgroup of G and a single element of G, then A has a generator.

Proof. We prove the first statement. Let $a \in A$ generate the finite cyclic quotient. Let n be the order of this quotient. Then $a^n \in T$. Let $g \in T$ be such that $a^n g$ generates T. Since T is divisible, there is $h \in T$ with $h^n = g$. Then the element ha generates the finite cyclic quotient and $(ha)^n$ generates T. The first statement follows.

Suppose that $A \subset G$ is the closure of an abelian subgroup of G generated by a connected subgroup R and an element of $g \in G$. The component group of A is finite. Let $B \subset A$ be the union of the connected components of A that contain a power of g. Then B is a closed subgroup of A that contains both R and g. This means $B = A$ and g generates the component group, implying that the component group is cyclic. Since the component of the identity of A is a compact, connected abelian Lie group, it is a torus. The result now follows from the first statement. □

Corollary 2.5. *1. Let A be an abelian subgroup of G containing a torus with cyclic quotient. Then A is contained in a maximal torus.*
2. If $A \subset G$ be an abelian group generated by a connected abelian group A_0 and a single element g, then A is contained in a maximal torus.

Proof. According to Corollary 2.4, in either case the closure of A has a generator. That generator is contained in a maximal torus and hence so is the closure of A. □

3 Maximal Tori in a Compact, Connected Lie Group

3.1 Definition and Existence

Let G be a non-trivial, compact connected lie group.

Proposition 3.1. *G contains a positive dimensional torus.*

Proof. Since G is connected and non-trivial it is positive dimensional. Thus, its Lie algebra is non-zero. Fix $X \neq 0$ in $\mathfrak{t}$. Then $\exp(tX)$ is a non-trivial one-parameter subgroup $A \subset G$. The group A is connected, positive dimensional and abelian. So is its closure, which is a Lie subgroup according to Theorem 3.9 of Lecture 2. By definition, this subgroup is a positive dimensional torus. □

Corollary 3.2. *There is a positive dimensional torus in G that is not properly contained in any other torus in G.*

Proof. We have seen that G contains a positive dimensional torus. Let T be a torus of maximal dimension in G. Then T is not properly contained in any other torus in G. For, if T is properly contained in a torus T', then since T' is connected, the Lie algebra of T' is strictly larger than that of T. This means that the dimension of T' is larger than the dimension of T. □

Definition 3.3. Any torus satisfying the conclusion of the previous claim is a *maximal torus.*

3.2 The Roots of a Maximal torus

Lemma 3.4. *1. If T is a maximal torus, then its Lie algebra $\mathfrak{t}$ is not properly contained in an abelian sub Lie algebra of $\mathfrak{g}$.*

2. If $g \in T$ is a topological generator of T, then the Lie algebra of the centralizer $Z(g)$ of g is $\mathfrak{t}$.

Proof. Suppose that L is an abelian subalgebra of $\mathfrak{g}$ properly containing $\mathfrak{t}$. The image $\exp(L)$ is a connected abelian subgroup containing T that contains a submanifold whose tangent space at e is L. The same is true of its closure, which is a closed, connected abelian subgroup and hence a torus. This torus properly contains T, which is a contradiction. This proves 1.

Let g be a topological generator of a the maximal torus T. Then the centralizer, $Z(g)$, of g is a Lie subgroup containing T and commuting with T. Let L be its Lie algebra. Then $\mathfrak{t} \subset L$ since $T \subset Z(g)$. Suppose this is a proper inclusion. Let $X \in L$ be an element not contained in $\mathfrak{t}$. Since $\exp(X)$ commutes with T, it follows that $[X, \mathfrak{t}] = 0$. Of course $[X, X] = 0$. Thus, $\mathfrak{t} \oplus \langle X \rangle$ is an abelian Lie algebra properly containing $\mathfrak{t}$. This contradicts the first item. $\square$

Applying the discussion of Section 1.3 and Part 2 of Lemma 3.4, we have the following.

Theorem 3.5. *The action of a maximal torus T decomposes $\mathfrak{g}$ as*

$$\mathfrak{g} = \mathfrak{t} \oplus V_1 \oplus \cdots \oplus V_r$$

where each V_i is two-dimensional and on which T acts by a non-trivial character $\alpha_i \colon T \to S^1$ followed by a standard semi-free rotation action of the circle on V_i.

Proof. By the results in Section 1.3 we need only how that 0-eigenspace, E_0, for the adjoint action of T on $\mathfrak{g}$ is contained in the Lie algebra $\mathfrak{t}$ of T. Since the adjoint action of T on E_0 is trivial, T commutes with $\exp(E_0)$. Let g be a topological generate of T. Then $\exp(E_0)$ is a subgroup of $Z(g)$, and hence E_0 is a subspace of the Lie algebra of $Z(g)$. Part 2 of Lemma 3.4 tells us that the Lie algebra of $Z(g)$ is $\mathfrak{t}$. $\square$

Remark 3.6. The characters $\alpha_i \colon T \to S^1$ are only defined up to sign, since reversing the orientation of V_i replaces α_i by $-\alpha_i$.

Definition 3.7. The non-zero weights of the action of T on $\mathfrak{g}$; i.e., the non-trivial characters $\{\pm\alpha_i\}_i$ of the action of T on $\mathfrak{g}$, are the *roots* of (G, T), or simply the roots of G if T is clear from context. The associated two-dimensional subspaces $V_i \subset \mathfrak{g}$ are the *root spaces*, with V_i being the root space for $\pm\alpha_i$.

4 The Weyl group

4.1 Definition and First results

Lemma 4.1. *The automorphism group of a torus is a discrete group.*

Proof. An automorphism of a torus T, lifts to a linear automorphism of its Lie algebra $\mathfrak{t}$ which stabilizes the kernel, Λ, of the exponential map. Since a linear isomorphism of a vector space that fixes the lattice Λ point-wise is the identity, we have an embedding of $\mathrm{Auto}(T) \subset \mathrm{Auto}(\Lambda)$. [It is easy to see that these automorphism groups are equal.] Then $\mathrm{Auto}(\Lambda) \subset GL(\mathfrak{t})$ is the subgroup stabilizing Λ. As such, it is a topologically closed subgroup of $GL(\mathfrak{t})$ and hence a Lie subgroup. There are only countable many automorphisms of a lattice, and hence this Lie group has only countably many elements. That is to say its Lie algebra is zero-dimensional and hence the Lie group is a discrete group. $\square$

Definition 4.2. Let T be a maximal torus of a compact, connected Lie group G. The *Weyl group* $W(G,T)$ of T is defined to be the quotient of the normalizer $N_G(T)$ of T in G by T:

$$W(G,T) = N_G(T)/T.$$

Proposition 4.3. *Let T be a maximal torus of a compact, connected Lie group G. The Weyl group of $W(G,T)$ is finite and is the component group of $N_G(T)/T$.*

Proof. By definition $N_G(T)$ is a topologically closed subgroup of G, hence it is a sub Lie group and its component group is finite. Let $N_0(T)$ be the component of the identity of $N_G(T)$. First of all we have a surjection $W(T) = N(T)/T \to N(T)/N_0(T)$ with kernel $N_0(T)/T$. The proposition follows once we show that $N_0(T) = T$.

We suppose that $N_0(T)$ properly contains T and deduce a contraction. Since T and $N_0(T)$ are connected, the Lie algebra $\mathfrak{t}$ of T is properly contained in the Lie algebra of $N_0(T)$. Choose a X in the Lie algebra of $N_0(T)$ that is not contained in $\mathfrak{t}$. Since the automorphism group of the torus is discrete, the adjoint action of the component of the identity $N_0(T)$ on T is trivial, and consequently the adjoint action of the Lie algebra of $N_0(T)$, and in particular the adjoint action of X, on $\mathfrak{t}$ is trivial. Of course, $[X, X] = 0$. Thus, the subspace V of $\mathfrak{g}$ spanned $\mathfrak{t}$ and X is an abelian Lie subalgebra properly containing $\mathfrak{t}$. This contradicts Part 1 of Lemma 3.4.. $\square$

4.2 The Adjoint Action of the Weyl Group on T.

Definition 4.4. Let T be a maximal torus of G. We define the Weyl group action on T, $W(G,T) \times T \to T$ to be given by $\overline{w} \cdot g = wgw^{-1} = Ad(w)(g)$ where $g \in T$ and $w \in N(T)$ with $\overline{w}$ as its image under the quotient map $N(T) \to N(T)/T) = W(T)$.

We define the Weyl group action on $T^* = \mathrm{Hom}(T, S^1)$ by $\overline{w} \cdot \varphi = \varphi \circ Ad(w^{-1})$;, where as before $w \in N(T)$ is a lift of $\overline{w}$.

Remark 4.5. Since the adjoint action of T on itself is trivial the adjoint action of $N_G(T)$ on T factors through the quotient $W(G,T) = N_G(T)/T$. The same is true for the action of $W(G,T)$ on T^*.

Notice for $\varphi \in T^*$ and $g \in T$ we have

$$(\overline{w} \cdot \varphi)(\overline{w} \cdot g) = \varphi(Ad(w)^{-1}(Ad(w)(g)) = \varphi(g).$$

That is to say the natural pairing $T^* \otimes T \to S^1$ is invariant under the Weyl actions.

Remark 4.6. For $w \in N(T)$ and $g \in T$ we have to distinguish between the product $wg \in G$ and the action $w \cdot g = \mathrm{Ad}(w)(g) = wgw^{-1}$. We always write the first by juxtaposition and the second with a $\cdot$.

Proposition 4.7. *Let G be a compact, connected Lie group and $T \subset G$ a maximal torus. The Weyl group action on T^* preserves the set of roots of T.*

Proof. First two general comments. Suppose that $\mu\colon T \to T$ is a Lie group isomorphism. Its action on T^* is given by $\mu \cdot \rho = \rho \circ \mu^{-1}$. Thus, the action of μ on T^* sends the roots of ad to the characters $\{\mu^*\alpha = \alpha \circ \mu^{-1}\}_\alpha$ as α ranges over the roots of the adjoint representation. Of course, conjugate representations have the same set of roots. Thus, if $\mathrm{ad} \circ (\mu^{-1})$ and ad are conjugate, then μ^* preserves the set of roots.

Now let us consider our case. Fix $w \in N_G(T)$ and $g \in T$ and $X \in \mathfrak{g}$.

Claim 4.8.

$$\mathrm{ad}(w^{-1} \cdot g)(X) = \mathrm{ad}(w^{-1}) \circ \mathrm{ad}(g) \circ \mathrm{ad}(w)(X).$$

Proof. Let $\gamma(t)$ be a smooth curve in G with $\gamma(0)=e$ and $\gamma'(0)=X$. Then

$$\mathrm{Ad}(w^{-1}\cdot g)(\gamma(t)) = w^{-1}gw(\gamma(t))w^{-1}g^{-1}w = w^{-1}\Big(g\big(w\gamma(t)w^{-1}\big)g^{-1}\Big)w.$$

Differentiating at $t=0$ gives the result. $\square$

We can reformulate this as a commutative diagram for $g \in T$ and the image $w \in N_G(T)$ we have

$$\begin{array}{ccc} T\times\mathfrak{g} & \xrightarrow{\mathrm{ad}(w^{-1}\cdot g)} & \mathfrak{g} \\ {\scriptstyle \mathrm{Id}_T\times\mathrm{ad}(w)}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}(w)} \\ T\times\mathfrak{g} & \xrightarrow{\mathrm{ad}(g)} & \mathfrak{g}. \end{array}$$

This is the statement that $\mathrm{ad}\colon T\to \mathrm{Auto}(\mathfrak{g})$ is conjugate to the representation given by

$$T \xrightarrow{w^{-1}\cdot} T \xrightarrow{\mathrm{ad}} \mathrm{Auto}(\mathfrak{g}).$$

But $\mathrm{ad}\circ(w^{-1}\cdot) = \overline{w}\mathrm{ad}$, where $\overline{w}\in W(G,T)$ is the image of $w\in N_G(T)$. This proves that $\overline{w}\mathrm{ad}$ is conjugate to ad and hence they have the same characters. Consequently, the action of $W(G,T)$ on T leaves invariant the set of roots of ad. $\square$

5 All Maximal Tori Are Conjugate and They Cover G

Theorem 5.1. *Let G be a compact, connected Lie group. Let $T\subset G$ be a maximal torus. Then every point $g\in G$ is contained in a conjugate of T. All maximal tori of G are conjugate.*

Proof. Let $g\in G$. Then $g\in xTx^{-1}$ if and only if $g(xT)=xT$. Said another way, $g\in G$ is in a conjugate of T if and only if, under the natural left action of $G\times G/T\to G/T$, the element g has a fixed point.

Lefschetz theory tells us that if $f\colon M\to M$ is a continuous self-map of a closed, oriented manifold and if $L(f)=\sum_i(-1)^i\mathrm{Trace}(f_*\colon H_i(M;Q)\to H_i(M;Q))$ is non-zero, then f has a fixed point. Of course, $L(f)$ depends only on the homotopy class of f. The number $L(f)$ is the homological intersection $[\Gamma(f)]\cdot[\Delta]$ in $M\times M$, where $\Gamma(f)$ is the embedding of $M\to M\times M$ as the graph of f and Δ is the embedding of M as the diagonal in $M\times M$.

If f is smooth and at each fixed point x of f, the differential of f at x does not have 1 as an eigenvalue, then we can say much more, Under these assumptions the graph $\Gamma(f)\colon M \to M \times M$ is transverse to Δ and $L(f)$ is the sum of local intersection numbers of $\Gamma(f)$ with Δ at the points of intersection. The intersection points are $\{(x,x)\}$ with x a fixed point of f, and the local intersection number at any such (x,x) is ± 1 and is equal to $\mathrm{sign}(\mathrm{Id} - \det(df_*(x)))$ as a map of $TM_x \to TM_x$.

Claim 5.2. *Let g_0 be a topological generator for T. The map $G \to G$ given by $h \mapsto g_0 h g_0^{-1}$ factors to given the map left multiplication by g_0 from G/T to G/T*

Proof. Since $g_0 \in T$, we have $g_0 h g_0^{-1} T = g_0 h T$. □

Let us compute the $L(g_0)$ for g_0 a topological generator of T. First, if $g_0 \in xTx^{-1}$, then since, the cyclic group generated by g_0 is dense in T, it follows that $T = xTx^{-1}$, meaning that $x \in N(T)$. Conversely, if $x \in N(T)$ then $T = xTx^{-1}$ and $g_0 \in xTx^{-1}$. Thus, the fixed points of the action of left multiplication by g_0 on G/T are the finite set $W(T) = N_G(T)/T$.

Let us compute the local Lefschetz number of g_0 at $[eT] \in G/T$. We have seen that the adjoint action of T on $\mathfrak{g}$ is given by $\mathfrak{t} \oplus V_0 \oplus_{i=1}^r V_i$ where T acts trivially on $\mathfrak{t}$ and by a non-trivial characters, $\pm\alpha_i$, on the two dimensional spaces V_i. The tangent space of G/T at $[T]$ is identified with $\oplus_{i=1}^r V_i$. By the claim above, left multiplication by g_0 on G/T agrees with the map induced on G/T by the action of $\mathrm{Ad}(g_0)$ on G. Hence, the differential of action of g_0 on $T_{[eT]}(G/T)$ is the restriction of the adjoint action to $\mathrm{ad}(g_0)$ to $\oplus_{i=1}^r V_i$ That is to say, $D(g_0\cdot)_*$ acting on $T_{eT}(G/T) = \oplus_{i=1}^r V_i$ preserves the direct sum decomposition and acts on V_i by

$$\begin{pmatrix} \cos(\alpha_i(g_0) & -\sin(\alpha_i(g_0)) \\ \sin(\alpha_i(g_0) & \cos(\alpha_i(g_0)) \end{pmatrix}$$

Thus

$$\det(\mathrm{Id} - Df_*(g_0\cdot)) = \prod_{i=1}^r (2 - 2\cos(\alpha_i(g_0))).$$

Since g_0 is a topological generator of T, each non-trivial character of T is non-trivial on g_0, so that each of these two-by-two matrices has positive determenent. Hence, the graph of the action of g_0 is transverse to the diagonal and the local intersection number is a sign, $+1$.

Now let $w \in N(T)$. Left multiplication by $w\colon G/T \to G/T$ sends $[w^{-1}T]$ to $[eT]$ and conjugates the left multiplication of g_0 at $[wT]$ to left multiplication of wg_0w^{-1} at $[eT]$. Thus, the local Lefschetz number of left multiplication by g_0 at the fixed point $[w^{-1}T]$ is the same as the local Lefschetz number of left multiplication of wg_0w^{-1} at $[eT]$. Since wg_0w^{-1} is a topological generator for T, the computation above applies to show that the local Lefschetz number of left multiplication by wg_0w^{-1} at $[eT]$ is 1. This shows that its local intersection number for left multiplication by g_0 at $w^{-1}T \in G/T$ is $+1$. This is true for every $w \in W(G,T)$ and consequently, the intersection number of the graph $\Gamma(g_0\cdot)$ with Δ, which is the Lefschetz number of multiplication by g_0 on G/T, is equal to $\#W(G,T) > 0$.

Now consider an arbitrary $g \in G$. Since G is connected, g and g_0 are connected by a path. Thus, left multiplication by g and g_0 on G/T are homotopic, and hence these actions on G/T have the same Lefschetz number, which we have just seen is non-zero. It follows that $g\colon G/T \to G/T$ has a fixed point $[xT]$, meaning that $g \in xTx^{-1}$. This proves that the conjugates of T cover G. Of course, each of these conjugates is a maximal torus.

Now let T' be another maximal torus of G and let g be a generator for T'. Then g is contained in a conjugate xTx^{-1}. Since g generates T', it follows that $T' \subset xTx^{-1}$. Since T' is maximal, $T' = xTx^{-1}$. This proves that all maximal tori are conjugate. □

Definition 5.3. The dimension of a maximal torus T in G is the *rank* of G.

Since all maximal tori of a compact, connected Lie group are conjugate, the maximal torus with its adjoint action on $\mathfrak{g}$ is, up to isomorphism independent of the choice of maximal torus. In particular, the roots of a maximal torus are, up to isomorphism, independent of the choice of maximal torus.

5.1 Consequences

Corollary 5.4. *The center of a compact, connected Lie is contained in every maximal torus.*

Proof. Let z be a central element of G. Then z is contained in a maximal torus T. Since every maximal torus is conjugate to T, every maximal torus contains z. This shows that every element of the center is contained in every maximal torus. □

Corollary 5.5. *For any compact, connected Lie group G, the exponential map* $\exp\colon \mathfrak{g} \to G$ *is onto.*

Proof. Let $T \subset G$ be a maximal torus. Then the Lie algebra $\mathfrak{t}$ is an abelian Lie algebra and the exponential map is surjective homomorphism from $(\mathfrak{t}, +0)$ to T. Thus, T is in the image of exp. We have just seen that every $g \in G$ is contain in a maximal torus. $\square$

Theorem 5.6. *Let T be a maximal torus in a compact, connected Lie group G. Then the action of $W(T) \times T \to T$ is effective.*

Proof. Let $Z(T) \subset N(T)$ be the centralizer of T. The statement in the proposition is equivalent to the statement that $Z(T) = T$. So suppose there is an element $z \in Z(T) \setminus T$. We have already seen that $W(T)$ is the component group of $N(T)$. This implies that $\{z\} \cup T$ generates an abelian group A containing T as a subgroup with finite cyclic quotient. According to Corollary 2.4 there is a generator a for A. The element a is contained in a maximal torus, T'. Since a generates A, $A \subset T'$, and *a fortiori* $T \subset T'$ Since $z \in T' \setminus T$, $T \neq T'$. This contradicts the fact that T is a maximal torus. $\square$

Definition 5.7. We have defined an effective action of $W(T)$ on T. Taking the differential of this action at the identity gives us an action $W(T) \times \mathfrak{t} \to \mathfrak{t}$. This is the quotient of the action $\mathrm{ad}\colon N(T) \to \mathrm{Auto}(\mathfrak{t})$.

Corollary 5.8. *The action $W(T)\colon \mathrm{Auto}(\mathfrak{t})$ is effective and leaves invariant the lattice $\Lambda \subset \mathfrak{t}$ that is the kernel of $\exp\colon \mathfrak{t} \to T$*

Proof. Since conjugation by $N(T)$ commutes with the exponential mapping which is a surjective homomoprhism with kernel Λ both the statements in the corollary are immediate from Theorem 5.6 and the fact that the action of $T \subset N(T)$ is trivial and the action of $W(T)$ is the one induced on the quotient $W(T) = N(T)/T$. $\square$