

Lie Groups: Fall, 2025

Lecture VI: Real Analyticity for Real Lie Groups

October 7, 2025

1 BCH Formula Redux

1.1 Iterated Lie Brackets and Polynomials

Suppose that L is a K -Lie algebra and $\{X_1, \dots, X_r\}$ is a K -basis for the underlying vector space. Then the basis determines coordinate functions $x^1, \dots, x^r: L \rightarrow K$. Namely, for any $X \in L$, $x^i(X)$ are defined so that $X = \sum_i x^i(X) X_i$.

The Lie algebra structure is given by constants $c_{i,j}^k$ defined by $[X_i, X_j] = \sum_k c_{i,j}^k X_k$. These are the *structure constants* of the Lie algebra in this basis. Skew symmetry of the Lie bracket is the statement that $c_{i,j}^k = -c_{j,i}^k$, i.e., that the $c_{i,j}^k$ are skew symmetric in i and j . The Jacobi identity is equivalent to

$$\sum_{\ell} (c_{i,\ell}^m c_{j,k}^{\ell} + c_{k,\ell}^m c_{i,j}^{\ell} + c_{j,\ell}^m c_{k,i}^{\ell}) = 0,$$

for all i, j, k, m . [CHECK THIS.]

Thus, given $X = \sum_i x^i X_i$ and $Y = \sum_j y^j X_j$ we have

$$[X, Y] = \sum_{r=1}^k \sum_{i,j} x^i y^j c_{i,j}^k X_k,$$

with coefficients of the X_k on the right-hand side being homogeneous polynomial functions of degree 1 in the x variables, $(x^1, \dots, x^r)$, and degree 1 in the y , variables $(y^1, \dots, y^r)$. The coefficients of these polynomial are themselves linear polynomials in the structure constants $c_{i,j}^k$. It is easy to see by induction that any iterated bracket in r copies of X and s copies of Y is $\sum_{k=1}^r f^k(x, y) X_k$ where each $f^k(x, y)$ is homogeneous polynomial function

of by degree r in the x variables and degree s in the y variables, and the coefficients of polynomial $f^k(x, Y)$ are polynomials of degree $(r + s - 1)$ in the $c_{i,j}^k$.

Recall that in the Hausdorff series $H(X, Y) = \sum_n H_n(X, Y)$, the term $H_n(X, Y)$ is a linear combination of iterated n -fold brackets of copies of X and Y . Thus $H_n(X, Y) = \sum_{k=1}^r f_n^k(x, y) X_k$ where f_n^k is homogeneous polynomial of degree n in the variables $(x^1, \dots, y^r)$. Thus, $H_n(X, Y)$ is an expression of the form $F_n(x, y)$ where F_n is a homogeneous polynomial function of degree n in $(x^1, \dots, y^r)$ with values in L . This shows that $H(X, Y)$ is a power series in the linear variables $(x^1, \dots, x^r, y^1, \dots, y^r)$ on $L \times L$ centered at $(0, 0)$ with values in L .

1.2 LOG and EXP

Suppose that A is an associative algebra and $X \in A$. Denote by $\text{EXP}(X)$ the power series $1 + \sum_{k=1}^{\infty} \frac{X^k}{k!}$. Similarly, denote by $\text{LOG}(1+X) = \sum_{k \geq 1} \frac{(-1)^{n-1} X^n}{n}$. At this point we are not considering whether the series converges or not, these are simply formal expressions.

The consequence of the last lecture is that if $S = \{X, Y\}$ the two power series $\text{EXP}(X)\text{EXP}(Y)$ and $H(X, Y) = \text{LOG}(\text{EXP}(X)\text{EXP}(Y))$ defined on $\widehat{U(FL)}(S) \times \widehat{U(FL)}(S)$ with values in $\widehat{U(FL)}(S)$ converge. The latter, $H(X, Y)$ is given as series whose n^{th} term is a linear combination iterated brackets of copies of X and Y . Since we are working in $\widehat{U(FL)}(S)$ the equality of power series simply means that for each N each of the series has only finitely many terms of degree $\leq N$ and these terms are the same.

1.3 BCH Formula and Local Lie Groups

Now consider a finite-dimensional, real Lie algebra L . Fix a positive definite inner product on this Lie algebra. the real vector space structure on L determines a real analytic structure.

Your homework problems were to show that there is a neighborhood U_L of $0 \in L$ with the property for $(X, Y) \in U_L \times U_L$ the series $\sum_n H_n(X, Y)$ converges uniformly and absolutely in L with limit a real analytic function

$$m_L: U_L \times U_L \rightarrow L.$$

We set $\Omega_L = m_L^{-1}(U_L) \cap (U_L \times U_L)$ with its induced real analytic structure so that m_L defines a real analytic map

$$m_L: \Omega_L \rightarrow U_L.$$

In fact, this map is given by a power series on $L \times L$ centered at $(0, 0)$ with values in L that is absolutely convergent on $\Omega_L \subset L \times L$. The identities $H(0, X) = H(X, 0) = X$ of formal power series imply that

$$m_L(0, X) = m_L(X, 0) = X \quad \text{for all } X \in U_L. \quad (1.1)$$

Similarly, the identity $H(X, -X) = 0$ implies

$$m_L(X, -X) = 0 \quad \text{for every } X \in U_L. \quad (1.2)$$

The identity $H(H(X, Y), Z) = H(X, H(Y, Z))$ of power series in three variables implies that

$$m_L(m_L((X, Y), Z)) = m_L(X, m_L(Y, Z)) \quad (1.3)$$

if $(m_L(X, Y), Z)$ and $(X, m_L(Y, Z))$ are contained in Ω_L

Definition 1.1. A *local Lie group structure* in a finite-dimensional vector space L consists of a five-tuple

$$(U_L, 0, -1, \Omega_L, m_L)$$

where U_L is open set of L containing 0; $\Omega_L \subset L \times L$ is an open neighborhood of $(0, 0)$; $m_L: \Omega_L \rightarrow U_L$ is a smooth map satisfying Equations (1.1–1.3).

In summary, the results of the previous lecture and the convergence results from your homework show that, for any real Lie algebra, the Hausdorff series defines a local Lie group in L . In fact, it is a real analytic local Lie group in the sense that m_L is given by a power series that is convergent on Ω_L and inverse is a real analytic function.

The germ of this local Lie group at 0 is independent of the choice of U_L with the property that the power series $H(X, Y)$ converges absolutely on $U_L \times U_L$ and is functorial in the Lie algebra L .

2 The case of $\mathfrak{gl}(n, \mathbb{R})$

We just defined an abstract local Lie group determined by any finite-dimensional, real Lie algebra. But more is true when the Lie algebra is $\mathfrak{gl}(n, \mathbb{R})$.

Claim 2.1. *The functions $X \mapsto \{X_{i,j}\}_{1 \leq i,j \leq n}$ give an identification of $M(n \times n)$ with $\mathbb{R}^{n^2}$. There is a ball $V \subset M(n \times n) = \mathbb{R}^{n^2}$ centered at $e \in GL(n, \mathbb{R})$ such that:*

1. $V \subset GL(n, \mathbb{R}) \subset M(n \times n)$.

2. There is an open neighborhood U of 0 in $\mathfrak{gl}(n, \mathbb{R})$ with $\exp|_U: U \rightarrow V$ being a diffeomorphism.
3. Writing any $g \in V \subset GL(n, \mathbb{R}) \subset M(n \times n)$ as $g = \text{Id} + A$ for $A \in M(n \times n)$ the power series

$$\text{LOG}(\text{Id} + A) = \sum_{n \geq 1} \frac{(-1)^{(n-1)} A^n}{n}$$

converges absolutely, uniformly on V .

4. The function, also denoted LOG , given by the power series in Item 3 is the inverse of $\exp|_U: U \rightarrow V$.

Proof. The existence of a neighborhood V as stated in the first three items is clear. Now consider $\text{Id} + A \in V$. Then, for $-1 \leq t \leq 1$, the arc $\text{Id} + tA$ is contained in V . Restricting to this interval we see that $\text{EXP}(\text{LOG}(\text{Id} + tA))$ is equal to Id at $t = 0$. Because the power series LOG and the power series EXP are formally inverses of each other, the functions $\text{Id} + tA$ and $\exp(\text{LOG}(\text{Id} + tA))$ on the interval agree to all orders at $t = 0$. This implies that the analytic functions are equal on the entire interval. Since this is true for an arbitrary $(\text{Id} + A) \in V$, the analytic function on V defined by the power series LOG is the inverse to $\exp|_U$. $\square$

Denote by $U_{\mathfrak{gl}(n, \mathbb{R})}$, $\Omega_{\mathfrak{gl}(n, \mathbb{R})}$ and $m_{\mathfrak{gl}(n, \mathbb{R})}$ the subsets and map described in the previous section for any Lie algebra for the special case of the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$. Without loss of generality, by taking $U_{\mathfrak{gl}(n, \mathbb{R})}$ smaller we can suppose that the map $(A, B) \mapsto \exp(A)\exp(B)$ sends $\Omega_{\mathfrak{gl}(n, \mathbb{R})} \rightarrow V$, where V is the open set V given in Claim 2.1.

Of course, the power series $\text{EXP}(A)$ converges absolutely and uniformly on every compact subset of $\mathfrak{gl}(n, \mathbb{R})$ to the exponential map $\exp: \mathfrak{gl}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}) \subset M(n \times n)$. Since matrix multiplication is given by polynomial expressions of the entries of the matrices being multiplied, the power series $\text{EXP}(A)\text{EXP}(B)$ centered at $(0, 0)$ that converges absolutely on $\mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R})$ to the function $\exp(X)\exp(Y)$. Since the image under this map of $\Omega_{\mathfrak{gl}(n, \mathbb{R})}$ is contained in V , it follows that $\text{LOG}(\text{EXP}(A)\text{EXP}(B))$ is a power series centered at $(0, 0)$ that converges absolutely on $\Omega_{\mathfrak{gl}(n, \mathbb{R})}$ to the function $(\exp|_{U_{\mathfrak{gl}(n, \mathbb{R})}})^{-1}(\exp(X)\exp(Y))$.

Set $S = \{X, Y\}$. As we noted above, for the free Lie algebra $FL(S)$ have an equality of power series series center at $(0, 0)$

$$H(X, Y) = \text{LOG}(\text{EXP}(X)\text{EXP}(Y)).$$

On the other hand, given elements $(A, B) \in \Omega_{\mathfrak{gl}(n, \mathbb{R})}$ there is a unique map of Lie algebras $FL(S) \rightarrow \mathfrak{gl}(n, \mathbb{R})$ sending $X \mapsto A$ and $Y \mapsto B$. Since $U(FL)(S)$ is the universal enveloping algebra of $FL(S)$ and $M(n \times n)$ is an associative algebra, this map induces an algebra map $U(FL)(S) \rightarrow M(n \times n)$ extending the given map $S \rightarrow M(n \times n)$. In particular it sends the power series $\text{LOG}(\text{EXP}(X)\text{EXP}(Y))$ to $\text{LOG}(\text{EXP}(A)\text{EXP}(B))$. It also sends the universal Hausdorff series $H(X, Y)$ to the Hausdorff series evaluated at (A, B) , namely $H(A, B)$.

Since $H(X, Y) = \text{LOG}(\text{EXP}(X)\text{EXP}(Y))$ as power series centered at $(0, 0) \in \widehat{U(FL)(S)} \times \widehat{U(FL)(S)}$, it follows that, as power series centered at $(0, 0)$ in $\mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R})$, we have

$$H(A, B) = \text{LOG}(\text{EXP}(A)\text{EXP}(B)).$$

But both series converge on $\Omega_{\mathfrak{gl}(n, \mathbb{R})} \subset \mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R})$. Thus, the functions on $\Omega_{\mathfrak{gl}(n, \mathbb{R})}$ that they converge to are equal. This means the function $m_{\mathfrak{gl}(n, \mathbb{R})} : \Omega_{\mathfrak{gl}(n, \mathbb{R})} \rightarrow U_{\mathfrak{gl}(n, \mathbb{R})}$ that is the limit of the Hausdorff series $H(A, B)$ is given by

$$m_{\mathfrak{gl}(n, \mathbb{R})}(A, B) = (\exp|_{U_{\mathfrak{gl}(n, \mathbb{R})}})^{-1}(\exp(A)\exp(B)).$$

We rewrite this as

$$\exp(m_{\mathfrak{gl}(n, \mathbb{R})}(A, B)) = \exp(A)\exp(B)$$

for $(A, B) \in \Omega_{\mathfrak{gl}(n, \mathbb{R})}$.

We have established the following.

Theorem 2.2. *Set $U_{\mathfrak{gl}(n, \mathbb{R})}$ and $\Omega_{\mathfrak{gl}(n, \mathbb{R})}$ and $m = m_{\mathfrak{gl}(n, \mathbb{R})}$ be the multiplication defined by the Hausdorff series $H(X, Y)$. Assume, as above, $U_{\mathfrak{gl}(n, \mathbb{R})}$ is chosen small enough that $\exp(X)\exp(Y) \in V$ for all $(X, Y) \in \Omega_{\mathfrak{gl}(n, \mathbb{R})}$, where V is as in Claim 2.1 The local Lie group $(U_{\mathfrak{gl}(n, \mathbb{R})}, 0, -1, \Omega_{\mathfrak{gl}(n, \mathbb{R})}, m_{\mathfrak{gl}(n, \mathbb{R})})$ is defined solely in terms of the Lie algebra structure of $\mathfrak{gl}(n, \mathbb{R})$. Let $\tilde{\Omega} \subset G \times G$ be the image $(\exp \times \exp)(\Omega_{\mathfrak{gl}(n, \mathbb{R})})$. Then there are commutative diagrams*

$$\begin{array}{ccc} \Omega_{\mathfrak{gl}(n, \mathbb{R})} & \xrightarrow{m_{\mathfrak{gl}(n, \mathbb{R})}} & U_{\mathfrak{gl}(n, \mathbb{R})} \\ \exp \times \exp \downarrow & & \downarrow \exp \\ \tilde{\Omega} & \xrightarrow{\text{multiplication}} & V \end{array}$$

and

$$\begin{array}{ccc} U_{\mathfrak{gl}(n, \mathbb{R})} & \xrightarrow{-1} & U_{\mathfrak{gl}(n, \mathbb{R})} \\ \exp \downarrow & & \downarrow \exp \\ V & \xrightarrow{\text{inverse}} & V. \end{array}$$

That is to say, the local Lie group $(U_{\mathfrak{gl}(n,\mathbb{R})}, 0, -1, m_{\mathfrak{gl}(n,\mathbb{R})})$ determined by the Lie algebra structure of $\mathfrak{gl}(n,\mathbb{R})$ is diffeomorphic, as a local Lie group, by $\exp$ to the local Lie group defined by a neighborhood of the identity in $GL(n,\mathbb{R})$.

As the next theorem shows, this is not only true for $\mathfrak{gl}(n,\mathbb{R})$ but also for any Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$.

Theorem 2.3. *Let $L \subset \mathfrak{gl}(n,\mathbb{R})$ be a Lie sub-algebra. Let $U_L = U_{\mathfrak{gl}(n,\mathbb{R})} \cap L$ and $\Omega_L = \Omega_{\mathfrak{gl}(n,\mathbb{R})} \cap (L \times L)$. Denote the restriction of m to Ω_L by m_L . The image of m_L is contained in U_L . There is a local Lie group*

$$(U_L, 0, -1, \Omega_L, m_L). \quad (*)$$

determined by the Lie algebra structure of L

As before m_L is given by the power series $H_L(X, Y) = \sum_{k \geq 0} (H_L)_k(X, Y)$ of iterated brackets of elements of L . This series converges absolutely on Ω_L .

Let $\mathcal{L} \rightarrow GL(n,\mathbb{R})$ be a map of Lie groups that is a one-one immersion with the image of the Lie algebra of $\mathcal{L}$ being L . Set $V_L = \exp(U_L) \subset \mathcal{L}$ and $\tilde{\Omega}_L = (\exp \times \exp)(\Omega_L)$. Then the exponential map identifies m_L with the restriction of group multiplication of $\mathcal{L}$ restricted to $\tilde{\Omega}_L$ and identifies $-1: U_L \rightarrow U_L$ with the inverse in $\mathcal{L}$ restricted to V_L .

Proof. We claim that $m_L(U_L \times U_L) \subset L$. The reason for this is that since L is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$ and hence all iterated brackets of element in L are contained in L , It follows that the terms $H_n(X, Y) \in L$ when $X, Y \in L$. The result then follows immediately from the result for $GL(n,\mathbb{R})$. $\square$

Now we need Ado's Theorem, which we shall not prove in this course

Theorem 2.4. *(Ado's Theorem) Every finite-dimensional, real Lie algebra has a faithful finite-dimensional, real linear representation, or equivalently, can be embedded in $\mathfrak{gl}(n,\mathbb{R})$ for some $n < \infty$.*

One consequence of Ado's Theorem is:

Theorem 2.5. *Let L be a finite-dimensional real Lie algebra. Then there is a Lie group G whose Lie algebra $\mathfrak{g}$ is isomorphic to L .*

Proof. Use Ado's Theorem to embed $L \subset \mathfrak{gl}(n,\mathbb{R})$ for some n . Then according to Lie's Theorems there is a Lie group G and a one-to-one immersion $G \rightarrow GL(n,\mathbb{R})$ whose differential at $e \in G$ identifies the Lie algebra $\mathfrak{g}$ of G with L . $\square$

As a corollary of Ado's Theorem and Theorem 2.3 we have:

Corollary 2.6. *Suppose that G is a Lie group with Lie algebra L . Choose a positive definite symmetric inner product on L with resulting norm $|\cdot|$. Then there is $\epsilon > 0$ such that setting $U_L \subset L$ equal to the ball of radius ϵ , the series $H(X, Y)$ converges uniformly and absolutely for X, Y in U_L . As before, we form a local Lie group*

$$(U_L, 0, -1, \Omega_L, m_L).$$

The germ of this local Lie group maps isomorphic via the exponential mapping to the restriction of multiplication and inverse of G restricted to the images of Ω_L and U_L under the exponential mapping.

Proof. By Ado's theorem there is an embedding $\mathfrak{g} \rightarrow \mathfrak{gl}(n, \mathbb{R})$ for some n . According to Lie's first theorem there is a map of Lie groups $G' \rightarrow GL(n, \mathbb{R})$ that is a one-one immersion so that the induced map on Lie algebras, $\mathfrak{g}' \rightarrow \mathfrak{gl}(n, \mathbb{R})$, maps $\mathfrak{g}'$ isomorphically onto $\mathfrak{g}$. Then the previous result tells us that the corollary as stated holds for G' replacing G .

But G' and G have the same Lie algebra and thus are isogenous. It follows immediately that there are open subset of the identity of G and of G' and an isomorphism between them that induces the given identification of their Lie algebras and identifies the multiplications and inverses and G and G' . Hence, possibly after replacing U_L with a smaller neighborhood of the identity, the corollary holds for G as well as for G' . $\square$

3 Real Analyticity of Real Lie groups

Theorem 3.1. *Every Lie group G inherits a natural real analytic structure from its Lie algebra via the exponential mapping that makes it a real analytic Lie group. Every Lie group homomorphism is real analytic with respect to these structures.*

Proof. Fix a Lie group G with multiplication μ and inverse ι . Let L be its Lie algebra. Then L has a neighborhood U_L of 0 invariant under $X \mapsto -X$ on which the series H converges to give an analytic function $m_L: \Omega_L \rightarrow U_L$. (in the real analytic structure on U_L coming from the vector space structure on L).

Possibly after replacing U_L by a smaller open set, the exponential map from $L \rightarrow G$ identifies U_L with an open subset $V \subset G$. The exponential map identifies the restriction of m_L to Ω_L and the restriction of -1 to U_L

with the restriction of $\mu: V \times V \rightarrow G$ to $\mu^{-1}(V)$. It also identifies -1 on U_L with $\iota: V \rightarrow V$.

Fix an open neighborhood $W \subset U_L$ of 0 , invariant under multiplication by -1 such that $(W \times m_L(W \times W)) \cup (m_L(W \times W) \times W) \subset \Omega_L$. Let $\overline{W} = \exp(W)$. The open set W inherits a real analytic structure from the linear structure on L . We use $\exp$ to transport this real analytic structure on W to a real analytic structure on $\overline{W}$, giving an analytic structure on a neighborhood of e in G . Now, we define a real analytic structure in a neighborhood $g\overline{W}$ of $g \in G$ by transporting the real analytic structure just defined on $\overline{W}$ via left multiplication by g . This gives a covering of G by real analytic patches.

To show that this defines a real analytic structure on G , we must check that on the overlap of two charts the two analytic structures agree. Suppose that $\overline{V} = g\overline{W} \cap h\overline{W} \neq \emptyset$. The overlap function from $g^{-1}\overline{V} \subset \overline{W}$ to $h^{-1}\overline{V} \subset \overline{W}$ is given by $h^{-1}g$. Set $V_0 \subset W$ and $V_1 \subset W$ be the pre-images under $\log$ of $g^{-1}\overline{V} \subset \overline{W}$ and $h^{-1}\overline{V} \subset \overline{W}$.

Fix $v_0, w_0 \in \overline{W}$ such that $g\overline{v}_0 = h\overline{w}_0$ with $\overline{v}_0 = \exp(v_0)$ and $\overline{w}_0 = \exp(w_0)$. Then $h^{-1}g = (\overline{w}_0)^{-1}\overline{v}_0$ and the overlap function $g^{-1}\overline{V} \rightarrow h^{-1}\overline{V}$ is given by multiplication by $h^{-1}g = (\overline{w}_0)^{-1}\overline{v}_0 = \exp(m_L(-w_0, v_0))$. Since $v_0, w_0 \in W$ and W is invariant under $X \mapsto -X$, we see that

$$m_L: \{m_L(-w_0, v_0)\} \times V_0 \rightarrow V_1$$

is a real analytic map. Thus, left multiplication by $(\overline{w}_0)^{-1}\overline{v}_0$ as a map $g^{-1}\overline{V}_0 \rightarrow h^{-1}\overline{V}_1$ is real analytic. This proves that the overlap function, multiplication by $h^{-1}g$, from $g^{-1}\overline{V} \rightarrow h^{-1}\overline{V}$ is real analytic.

This, and the symmetric argument for the inverse, show that any two of the given real analytic charts give the same real analytic structure on their overlap and hence the collection of analytic charts $\{g\overline{W}\}_{g \in G}$ defines an atlas giving a global real analytic structure on G . We call this the *real analytic structure generated by the BCH formula*.

Let us consider multiplication near $(g, h) \in G \times G$. The analytic structure on $g\overline{W} \times h\overline{W} \subset G \times G$ is given by $(g\overline{w}, h\overline{v})$ with the analytic structure on $\overline{w}$ and $\overline{v}$ as defined previously. The product is given by $g\overline{w}h\overline{v} = gh(h^{-1}\overline{w}h)\overline{v}$. Since we know the product is analytic on $\overline{W} \times \overline{W}$, we need only see that conjugation by h is analytic near the identity. But conjugation by h is a homomorphism of Lie groups whose map on the Lie algebras is a linear

isomorphism of Lie algebras. There is a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\text{ad}(h)} & L \\ \exp \downarrow & & \exp \downarrow \\ G & \xrightarrow{\text{Ad}(h)} & G. \end{array}$$

Since near $0 \in L$ and $e \in G$, $\exp$ transports the analytic structure determined by the vector space structure of L to the analytic structure on G determined by the BCH formula, it follows that near $e \in G$ conjugation by h is real analytic since $\text{ad}(h)$ is a linear map of L and thus obviously analytic.

Lastly, let us consider the inverse map. Near g it sends $g\bar{w}$ to $\bar{w}^{-1}g^{-1} = g^{-1}(g\bar{w}^{-1}g^{-1})$. For $\bar{w} \in \bar{W}$ the map $\bar{w} \mapsto (\bar{w})^{-1}$ lifted to W is the map $w \mapsto -w$, which is clearly real analytic. Thus $\bar{w} \mapsto (\bar{w})^{-1}$ is real analytic. We have already seen that conjugation by g is real analytic near e . It follows that the inverse map $\iota: G \rightarrow G$ is real analytic. This completes the proof that G with the analytic structure generated by the Baker-Campbell-Hausdorff formula is a real analytic Lie group.

Now suppose that $\psi: G \rightarrow H$ is a Lie group homomorphism. It induces a linear map $d_e\psi: \mathfrak{g} \rightarrow \mathfrak{h}$ which is linear and a map of Lie algebras. Thus, it determines an analytic map between local Lie groups in $\mathfrak{g}$ and $\mathfrak{h}$. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d_e\psi} & \mathfrak{h} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\psi} & H \end{array}$$

Since the real analytic structures near $e \in G$ and $e \in H$ are transported from the usual real analytic structure on $\mathfrak{g}$ and $\mathfrak{h}$ and since $d_e\psi$ is linear and hence real analytic, it follows that, near the identity in G , the map ψ is real analytic.

Now let us consider ψ in a neighborhood of $g \in G$. In a neighborhood of g , the map is given by $\psi(g\bar{w}) = \psi(g)\psi(\bar{w})$ for $\bar{w}$ near the identity in G . Since multiplication by g in G and multiplication by $\psi(g)$ in H are real analytic isomorphisms, and $\bar{w} \mapsto \psi(\bar{w})$ is a real analytic map near the identity by what we just observed, it follows that ψ is real analytic near g , and consequently, is a real analytic map.

This shows that the category of real analytic Lie groups is equivalent to the category of smooth Lie groups. (Assuming, of course, Ado's theorem.)

□

4 Summary of our Results About Lie Algebras

We have finished approximately three weeks of study of Lie algebras. Here, we recap what we have established.

We began with the question about linear representations of a real Lie algebra L on a vector space V . By this we mean a linear map $\rho: L \otimes V \rightarrow V$ that, for all $X, Y \in L$ and $v \in V$ satisfies

$$\rho(X \otimes \rho(Y \otimes v)) - \rho(Y \otimes \rho(X \otimes v)) = \rho([X, Y] \otimes v).$$

We define $\text{ad}(\rho): L \rightarrow \text{End}(V)$ by $\text{ad}(\rho)(X)(v) = \rho(X \otimes v)$. Then ρ is a representation of the Lie algebra L if and only if $\text{ad}(\rho): L \rightarrow \text{End}(V)$ is a map of Lie algebras. A very closely related notion is that of a map $\mu: L \rightarrow A$ where A is an associative algebra and μ is a homomorphism of Lie algebras when we give A the Lie algebra bracket $[x, y] = xy - yx$.

We found the universal solution to the second problem. It is given by a map $L \rightarrow U(L)$, where $U(L)$ is the universal enveloping algebra. To construct $U(L)$ we take the tensor algebra, $T(L)$, generated by L , and impose the relations $X \otimes Y - Y \otimes X = [X, Y]$ for $X, Y \in L$.

It is a straight-forward formal argument to show that this produces a universal solution to the problem of finding a Lie algebra map from L to the xy-yx Lie algebra underlying an associative algebra. More delicate is to understand the structure of $U(L)$. **The main theorem about the universal enveloping algebra, the Poincaré-Birkhoff-Witt Theorem, is that $U(L)$ has a natural increasing multiplicative filtration whose associated graded algebra is isomorphic to the polynomial algebra on L . In particular, the natural map $L \rightarrow U(L)$ is an inclusion.** This is the first major theorem of our study of Lie algebras. This theorem is proved by defining an action of L on the polynomial algebra $P(L)$ so that the action of $X \in L$ on a homogeneous polynomial is multiplication by X to highest order.

We also showed that $U(L)$ has a natural co-multiplication making it a bi-algebra (even a Hopf algebra). The second main result is deduced from the fact that the isomorphism from the associated graded algebra of $U(L)$ to the polynomial algebra on L identifies the associated graded version of the co-multiplication in $U(L)$ with the usual co-multiplication in $P(L)$. This leads to our second main result: **the primitive elements of the co-multiplication in $U(L)$ are exactly the elements of $L \subset U(L)$.** We apply this to show that for elements $X, Y \in L$ the power series $\log(\exp(X)\exp(Y))$ in the completion of $\widehat{U}(L)$ with respect to the ideal generated by L is equal to the Hausdorff series $H(X, Y)$ whose terms of degree

n are elements of L ; indeed, the n^{th} order term is a linear combination of iterated brackets of length n of copies of X and Y . **This series is the formal multiplication in a formal group on the Lie algebra. This is the Campbell-Baker-Hausdorff Formula.**

Using standard convergence results in your homework, you showed that, for a finite-dimensional, real Lie algebra L , the Hausdorff series converges absolutely and uniformly for X, Y in some neighborhood of 0. **The function defined by the Hausdorff series is a local group multiplication in the neighborhood of 0 in L . It, together with the local inverse, $X \mapsto -X$, defines a local group structure on a neighborhood of $0 \in L$, a local group structure that is functorial in the Lie algebras L .**

In the case of $L = \mathfrak{gl}(n, \mathbb{R})$, the exponential map produces an isomorphism between this local group structure in a neighborhood of 0 in $\mathfrak{gl}(n, \mathbb{R})$ to the local group structure given by $GL(n, \mathbb{R})$ restricted to a neighborhood of the identity.

It is easy to generalize this result to any sub-algebra of $\mathfrak{gl}(n, \mathbb{R})$. Ado's theorem (which we do not prove) that says that every finite-dimensional, real Lie algebra is isomorphic to a sub-algebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n . The final major result is:

- **Every finite-dimensional, real Lie algebra L is the Lie algebra of a Lie group $G(L)$.**
- **In a neighborhood of the identity the multiplication of $G(L)$ is given as the image under exponential map of the local multiplication in a neighborhood of $0 \in L$ given by the Hausdorff series.**

This construction produces a local real analytic structure near the identity of any Lie group in which multiplication and inverse are real analytic maps. It is easy to extend this real analytic structure near the identity by equivariance to the entire Lie group in such a way that multiplication and inverse are real analytic maps. This leads to a subcategory of the category of Lie groups, real analytic Lie groups. The subcategory of real analytic Lie groups is a fully faithful subcategory of the category of Lie groups. **This means that every Lie group is isomorphic, as a Lie group, to a real analytic Lie group, and every Lie group homomorphism between real analytic Lie groups is real analytic.**