Lie Groups: Fall, 2025
Lecture IV: The Universal Enveloping Algebra

September 30, 2025

1 Introduction

The Baker-Campbell-Hausdorff Theorem asserts the existence of a universal power series in two variables in a Lie algebra where the terms are sums iterated brackets of the two variables. For any finite dimensional real Lie algebra L, with a norm, this series converges absolutely in some ϵ-neighborhood of $(0,0)$ in $L \times L$. This series defines a local Lie group multiplication on some neighborhood of $0 \in L$. If $L = \mathfrak{g}$ for some Lie group G, the exponential mapping is an isomorphism between this local Lie group on a neighborhood of $0 \in \mathfrak{g}$ and a local Lie group determined by a neighborhood of $e \in G$.

In this lecture and the next, we prepare the way to prove this theorem by establishing it in a formal sense, working with power series completions. The homework carries out the appropriate estimates to show that for a finite dimensional real Lie algebra L, for $V \subset L$ a sufficiently small neighborhood of 0, the power series converge absolutely.

Let me begin the discussion by pointing out the issue. We write

$$\exp(a+b) = \sum_{k=0}^{\infty} \frac{(a+b)^k}{k!}.$$

If we are in an algebra where a and b do not commute, then $(a+b)^k$ is the sum of all 2^n strings of a's and b's of length n. The quadratic term is $a^2+ab+ba+b^2$ and the cubic term is $a^3+a^2b+aba+ba^2+ab^2+bab+ba^2+b^3$. It is only when a and b commute that we can rearrange these terms of degree k to produce

$$\sum_{m=0}^{k} \binom{k}{m} a^m b^{k-m}.$$

In this case the factorials cancel nicely to yield $\exp(a) \cdot \exp(b) = \exp(a + b)$.

In particular, in a non-commutative Lie algebra and Lie group we cannot assert that $\exp(X)\exp(Y) = \exp(X + Y)$. Of course, one can keep track of the switches by replacing YX by $XY - [X, Y]$, so it is not unreasonable that the product can be written as a power series using iterated brackets of X, Y.

Working by induction, one can write an explicit formula to any finite order. In fact there is a general formula which we establish in the problems. But there is a more conceptual way to approach the question using free Lie algebras and bi-algebra structure on their universal enveloping algebras. This lecture introduces the Universal Enveloping Algebra of a Lie Algebra.

2 PBW Theorem concerning the Universal Enveloping Algebra of a Lie Algebra

For this section we fix a field K of characteristic 0. For the rest of this lecture by an *associative algebra* we mean an associative K-algebra with unit; by a vector space will mean a K-vector space; and by a Lie algebra will mean a Lie algebra over K. . If V is a vector space then $\mathrm{End}(V)$ is a associative algebra with the multiplication being given by the composition of endomorphisms and the identity being the identity endomorphism. Hence, $\mathrm{End}(V)$ is a Lie algebra with $[A, B] = AB - BA$.

Definition 2.1. By a *linear representation* of a Lie algebra L on a vector space V we mean a Lie algebra map $\rho\colon L \to \mathrm{End}(V)$. Equivalently, we can define a linear representation of L on V as a bilinear map

$$\rho\colon L \otimes V \to V$$

satisfying $\rho(X)(\rho(Y)(v)) - \rho(Y)(\rho(X)(v)) = \rho([X, Y])(v)$ for all $X, Y \in L$ and all $v \in V$.

N.B. If $K = \mathbb{R}$ (or $\mathbb{C}$) and $\rho\colon G \times V \to V$ is a finite dimensional linear representation of a real (or complex) Lie group G, then the differential at the identity $D_e\rho\colon \mathfrak{g} \to End(V)$ is a linear representation of the Lie algebra $\mathfrak{g}$ of G.

If $\rho\colon L \otimes V \to V$ is a linear representation then the image $\rho(L) \subset \mathrm{End}(V)$ is a linear subspace but its image is not (usually) an associative sub-algebra of $\mathrm{End}(V)$. The image $\rho(L)$ is is however a sub-Lie algebra of the $AB - BA$ Lie algebra of an associative algebra. If ρ is an faithful representation ($\mathrm{Ker}(\rho) = 0$), then this embeds L as a sub-Lie algebra of the $AB - BA$ Lie algebra of an associative algebra.

The Poincaré-Birkhoff-Witt Theorem (PBW Theorem) says that every finite dimensional Lie algebra L is a sub Lie algebra of the Lie algebra given by an associative algebra with the bracket $[A, B] = AB - BA$. It does this by creating an infinite dimensional faithful linear representation, V, of L, and $U(L)$ is the associative sub-algebra of the endomorphism algebra of V generated by the image of L.

Thus, the construction produces an injection $\mu\colon L \to U(L)$ where $U(L)$ is an associative algebra and μ is a map of Lie algebras from L to the $AB - BA$ Lie algebra structure on $U(L)$. In fact, this construction turns out to be the universal solution to the problem of mapping L to the Lie algebra of an associative algebra. By this we mean that if $\rho\colon L \to A$ with A an associative algebra and ρ a morphism of Lie algebras, then there is a unique map $f\colon U(L) \to A$ of associative algebras with $\rho = f \circ \mu$.

The associative algebra $U(L)$ and the map $L \to U(L)$ of the construction is called the *Universal Enveloping Algebra* of the Lie algebra.

2.1 The Construction of the Universal Enveloping Algebra $U(L)$

Let $(L, [\cdot, \cdot])$ be a finite dimensional Lie algebra. Consider the tensor algebra

$$T(L) = \sum_{n=0}^{\infty} \otimes^n L$$

(tensor product over K) with the usual (associative) multiplication defined by juxtaposition of tensors. We denote by $T^k(L) = \otimes^k L$ so that $T(L) = \sum_{k \geq 0} T^k(L)$ is a graded algebra. There is the natural identifiation $L = T^1(L) \subset T(L)$. Then $T(L)$ is the free associative algebra generated by L in the sense that given an associative algebra A and a linear map $\psi\colon L \to A$ there is a unique extension of ψ to a map of associative algebras $T(L) \to A$.

We define the *universal enveloping algebra of* L, denoted $U(L)$, to be the quotient of $T(L)$ by the two-sided ideal generated by $(x \otimes y - y \otimes x - [x, y])$ for all $x, y \in L$ with the map $L \to U(L)$ being the composition of the natural map $L \to T^1(L) \subset T(L)$ followed by the quotient map $T(L) \to U(L)$.

By construction $U(L)$ inherits an associative algebra structure from that of $T(L)$, and the map $\mu\colon L \to U(L)$ is a homomorphism of Lie algebras when $U(L)$ is given the $AB - BA$ Lie bracket coming from its associative multiplication.

Proposition 2.2. *Given an associative algebra A and a map ρ of Lie algebras from L to the Lie algebra determined by the associative multiplication on A, there is a unique map of algebras $f\colon U(L) \to A$ with $f \circ \mu = \rho$. In particular, any linear representation of the Lie algebra $L \otimes V \to V$ extends to a unique associative algebra action $U(L) \otimes V \to V$.*

Proof. If we have an associative algebra A and a linear map $\rho\colon L \to A$ with $\rho([X,Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$ then by the universal property of $T(L)$ there is a unique map $\hat{\rho}\colon T(L) \to A$ of associative algebras extending ρ. Since ρ is a map of Lie algebras, $\hat{\rho}$ sends every defining relation for $U(L)$ to zero in A. Hence, $\hat{\rho}$ factors to define a map of associative algebras $U(L) \to A$. Since the image of L in $U(L)$ generates $U(L)$ is the unique map of associative algebras $U(L) \to A$ extending ρ. This establishes that $U(L)$ has the universality property stated in the proposition.

If $L \otimes V \to V$ is a representation, then the associated map $L \to \mathrm{End}(V)$ is a map of Lie algebras and the extension of this map to an algebra map $U(L) \to \mathrm{End}(V)$ follows as a special case of the general result. The algebra homomorphism $U(L) \to \mathrm{End}(V)$ is the same thing as an algebra action $U(L) \otimes V \to V$. □

Of course, we have not yet ruled out that the map $L \to U(L)$ has a non-zero kernel, or even that $U(L)$ could be 0 for some non-zero Lie algebra. By the universal property of $L \to U(L)$, any linear representation of L on a vector space V extends to a map of algebras $U(L) \to \mathrm{End}(V)$. So, if the map $L \to U(L)$ has a non-zero kernel L_0, this would mean that every linear representation of L vanishes on L_0.

This is where the PBW Theorem comes into play. It shows (among other things) that $L \to U(L)$ is an injection for every finite dimensional Lie algebra (over a field of characteristic 0).

2.2 The Statement

Theorem 2.3. *(PBW) Let L be a finite dimensional Lie algebra. There is a natural increasing filtration on the tensor algebra $T(L)$ defined by $F_n(T(L)) = \sum_{k=0}^{n} \otimes^k L$. We define an increasing filtration of $U(L)$ by setting $F_n(U(L))$ equal to the image of $F_n(T(L))$ under the quotient map $T(L) \to U(L)$. This is a multiplicative filtration in the sense that the multiplication induces a map $F_n(U(L)) \otimes F_m(U(L)) \mapsto F_{n+m}(U(L))$. Let*

$$Gr_*^F(U(L))) = \oplus_{n=0}^{\infty} F_n(U(L)/F_{n-1}(U(L))$$

be the associated graded algebra. The natural map map $L \to F_1(T(L)) \to F_1(U(L))$ induces with a linear map $\mu_L\colon L \to Gr_1^F(U(L))$. The map μ_L extends to an isomorphism of associative graded algebras from the polynomial algebra, $P(L)$, on L to $Gr_*^F(U(L))$. In particular, the natural map $\mu_L\colon L \to U(L)$ is an injection.

Corollary 2.4. *Every (finite dimensional) Lie algebra is a sub Lie algebra of a Lie algebra given by the $AB-BA$ Lie bracket of an associative algebra.*

Remark 2.5. If L is an abelian Lie algebra (meaning the bracket is identically zero), then $U(L)$ is the quotient of $T(L)$ by the two-sided ideal generated by $xy-yx$. In this case $U(L)$ is naturally isomorphic to the polynomial algebra $P(L)$.

2.3 The Proof of PBW Theorem

We fix a K-basis $\{X_i\}_{i\in I}$ for L and choose a total ordering for the basis, or equivalently a total ordering on the index set I. The images of the X_i in the polynomial algebra $P(L)$ generated by L are a multiplicative basis. To avoid confusion we use the notation z_i for the element in $P(L)$ determined by X_i, so that $P(L)$ is the algebra of all polynomials in $\{z_i\}_{i\in I}$.

For a finite sequence $J = \{j_1, \ldots, j_k\}$ of elements of I and $i \in I$, the notation $i \preceq J$ means that $i \leq j_r$ for all $j_r \in J$. We denote the number of elements in the sequence J by $|J|$. We denote by z_J the product $z_{j_1} \cdots z_{j_r}$ in $P(L)$. Of course, this element depends only on the set J not the ordering of its elements.

2.3.1 The Action and its inductive properties

Our goal is to define an action of the Lie algebra L on the vector space $P(L)$.

$$\sigma\colon L \otimes P(L) \to P(L);$$

that is to say a map of Lie algebras from $L \to \mathrm{End}(P(L))$ (where $\mathrm{End}(P(L))$ refers to the endomorphism algebra of the vector space $P(L)$). We do this by induction. Let $P^{\leq p}(L)$ denote the subspace of polynomials of degree $\leq p$. The inductive hypothesis for p is that we have a map $\sigma_p\colon L \otimes P^{\leq p}(L) \to P^{\leq (p+1)}(L)$ satisfying the following:

$A(p)$: If $i \preceq J$ for $|J| \leq p$, then $\sigma_p(X_i)z_J = z_i z_J$.

$B(p)$: For J with $|J| = q \leq p$ we have $\sigma_p(X_i)z_J - z_i z_J \in P(L)^q$.

$C(p)$: For J with $|J| < p$,

$$\sigma_p(X_i)\sigma_p(X_j)z_J - \sigma_p(X_j)\sigma_p(X_i)z_J = \sigma_p([X_i, X_j])z_J.$$

(Also, $\sigma_p|_{P^{\leq(p-1)}(L)} = \sigma_{p-1}$.)

For $p = 0$, we define $\sigma_0(X_i)1 = z_i$. Extending by K-linearity defines σ_0. By construction, σ_0 satisfies Condition $A(0)$. It satisfies Condition $B(0)$ since $B(0)$ follows from $A(0)$. Condition $C(0)$ is vacuous. This establishes σ_0 as required.

Now suppose that for some $p \geq 1$ we have defined $\sigma_{p-1} \colon L \otimes P^{\leq(p-1)}(L) \to P^{\leq p}(L)$ satisfying Conditions $A(p-1)$, $B(p-1)$, and $C(p-1)$. To extend σ_{p-1} to a map σ_p We need only define σ_p on all monomials z_J with $|J| = p$. If $i \preceq J$, we define $\sigma_p(X_i)z_J = z_i z_J$. Otherwise, re-odering J we have $J = (k, K)$ with $k \preceq K$ and $k < i$. By Condition $A(p-1)$ we have $z_J = \sigma_{p-1}(X_k)z_K$. Guided by the hope that Condition $C(p)$ will hold, we define

$$\sigma_p(X_i)z_J = \sigma_p(X_i)\sigma_{p-1}(X_k)z_K = z_k(\sigma_{p-1}(X_i)z_K) + \sigma_{p-1}([X_i, X_k])z_K.$$

Clearly, the right-hand side of this expression is defined by induction.

By linearity this defines the unique extension of σ_{p-1} to σ_p to linear map $L \otimes P^{\leq p}(L) \to P^{\leq(p+1)}(L)$. We must show this extension satisfies $A(p)$, $B(p)$, and $C(p)$.

Clearly, by construction Condition $A(p)$ holds for σ_p.

We check that Condition $B(p)$ holds. Clearly, it holds for $\sigma_p(X_i)z_J$ if $i \preceq J$. If not, we write $J = (k, K)$ with $k \preceq K$ and $k < i$.

$$\begin{aligned} \sigma_p(X_i)z_J &= \sigma_p(X_i)\sigma_{p-1}(X_k)Z_K \\ &= \sigma_p(X_k)\sigma_{p-1}(X_i)z_K + \sigma_{p-1}([X_i, X_k])z_K \end{aligned}$$

By Condition $B(p-1)$ there is $w \in P^{\leq p-1}(L)$ such that $\sigma_{p-1}(X_i)z_K = z_i z_K + w$. Plugging this equality into the last line above gives

$$\begin{aligned} \sigma_p(X_i)z_J &= \sigma_p(X_k)(z_i z_K + w) + \sigma_{p-1}([X_i, X_k])z_K \\ &= z_k z_i z_J + \sigma_{p-1}(X_k)(w) + \sigma_{p-1}([X_i, X_k])z_K \\ &= z_i z_K + \sigma_{p-1}(X_k)(w) + \sigma_{p-1}([X_i, X_k])z_K. \end{aligned}$$

Since $w \in P^{\leq(p-1)}(L)$, we have $\sigma_{p-1}(X_k)(w) \in L^{\leq p}(L)$. Also, $\sigma_{p-1}([X_i, X_k])z_K \in P^{\leq p}(L)$. This establishes Condition $B(p)$.

It remains to show that $C(p)$ holds. Consider a multi-index K with $|K| = p-1$. By construction $C(p)$ holds for $\sigma_p(X_i)\sigma_{p-1}(X_j)z_K$ if $j \preceq K$ and $i \geq j$. By symmetry it holds if $i \preceq K$ and $j \geq i$. We have established the following

Claim 2.6. *If $|K| < p$ or if $|K| = p$ and one of $i, j \preceq K$ then*

$$\sigma_p(X_i)\sigma_{p-1}(X_j)z_K - \sigma_p(X_j)\sigma_{p-1}(X_i)z_K = \sigma_{p-1}([X_i, X_j])z_K.$$

The remaining cases are where $K = (k, M)$ with $k \preceq M$ and $k < i, j$. To simplify the notation we drop σ_p and σ_{p-1} from the notation and simply write the product as a juxtaposition. We claim that

$$\begin{aligned} X_iX_jz_K = X_iX_jX_kz_M &= X_iX_kX_jz_M + X_i[X_j, X_k]z_M \\ &= X_kX_iX_jz_M + [X_i, X_k]X_jz_M + X_i[X_j, X_k]z_M \end{aligned}$$

The first equality is from the fact that $k \preceq M$ and $K = \{k\} \cup M$, and the second follows from the fact that $k \preceq M$ (or by the fact that $C(p-1)$ holds). To establish the third, we us the fact that $X_jz_M = z_jz_M + w$ for $w \in P^{\leq(p-2)}(L)$. Since $k \preceq \{j\} \cup M$ it follows from Claim 2.6 that

$$X_iX_k(z_jz_M) = X_kX_i(z_jz_M) + [X_i, X_k](z_jz_M).$$

Since $C(p-1)$ holds,

$$X_iX_kw = X_kX_iw + [X_i, X_k]w.$$

Together these two equations establish the third equality. By the anti-symmetric argument we have

$$X_jX_iz_K = X_jX_iX_kz_M = X_kX_jX_iz_M + [X_j, X_k]X_iz_M + X_j[X_i, X_k]z_M.$$

The first equation minus the second one yields:

$$\begin{aligned} X_iX_jz_K - X_jX_iz_K =& X_k[X_i, X_j]z_M + [[X_i, X_k], X_j]z_M + [X_i, [X_j, X_k]]z_M \\ =& (X_k[X_i, X_j] + [X_j, [X_k, X_i]] + [X_i, [X_j, X_k]]z_M. \end{aligned} \tag{2.1}$$

The Jacobi identity tells us that

$$[X_j, [X_k, X_i]] + [X_i, [X_j, X_k]] = -[X_k, [X_i, X_j]].$$

Thus, Eequation 2.1 becomes

$$\begin{aligned} X_iX_jz_K - X_jX_iz_K &= X_k([X_i, X_j]z_M - [X_k, [X_i, X_j]]z_M = [X_i, X_j]X_kZ_M \\ &= = [X_i, X_j]z_K. \end{aligned}$$

This completes the proof of property $C(p)$ and hence completes the inductive proof of the existence of the action $\sigma \colon L \otimes P(L) \to P(L)$ with properties $A(p), B(p), C(p)$ for all $p \geq 0$.

2.3.2 The graded isomorphisms

By Condition $C(p)$ for all p, the map resulting map $\sigma\colon L \to End(P(L))$ is a map of Lie algebras and hence extends to an algebra action $\sigma\colon U(L) \otimes P(L) \to P(L)$. By Condition $B(p)$ we have $\sigma(X_i)z_M = z_i z_M$ modulo $F_{|M|}P(L)$, and hence

$$\sigma(X_{i_1} \cdots X_{i_t})z_M = z_{i_1} \cdots z_{i_t} z_M \quad \text{modulo} \quad F_{|M|+t-1}U(L).$$

We define a vector space map $\varphi\colon U(L) \to P(L)$ be sending $a \in U(L)$ to $\varphi(a) = \sigma(a) \cdot 1$. Then $\varphi\colon U(L) \to P(L)$ is compatible with the gradings by degree. Of course, $P(L)$ is already a graded algebra and hence naturally isomorphic to its associated graded algebra. The associated graded $Gr^F\varphi$ induces a map of graded algebras $Gr^F\varphi\colon Gr^F_* U(L) \to P(L)$ sending the element $X_{i_1} \cdots X_{i_t}$ to the monomial $z_{i_1} \cdots z_{i_t}$. This shows that this is a surjective map of graded algebras.

We claim that the map of graded algebras is also injective. Since every element of $F_n U(L)$ is represented by a sum of monomials of degrees $\leq n$ in the X_i. Monomials of degree less than n and any two monomials of degree n that involve exactly the same X_i each the same number of times, just in different orders, are equal modulo $F_{n-1}U(L)$. It follows that $F_n U(L)/F_{n-1}U(L)$ is a quotient of the vector space generated by the monomials of length n given by weakly ordered sequences of the elements X_i. Since the weakly ordered sequences map via $Gr^F(\varphi)$ to a basis for the homogeneous polynomials of degree n, it follows that for each $n \geq 1$ these monomials of degree n are linearly independent and hence are a basis for $F_n U(L)/F_{n-1}U(L)$.

This shows that $Gr^F(\varphi)\colon Gr^F(U(L)) \to P(L)$ is an isomorphism of graded algebras. Hence, $\varphi\colon U(L) \to P(L)$ is a linear isomorphism of the filtered vector spaces that induces an isomorphism of the associated graded algebras.

Obviously, the composition of $Gr^F\varphi$ following the natural map $L \to F_1(U(L))$ is the identity from L to polynomials of degree 1 in L. Thus, the inverse of $Gr^F\varphi$ is an isomorphism of graded algebra $P(L) \to Gr^F_*(U(L))$ extending the given map $\mu\colon L \to Gr^F_1(U(L))$.

This completes the proof of the PBW Theorem.

2.3.3 A Further Consequence

Notice that the map $\varphi\colon U(L) \to P(L)$ given above is a filtered vector space isomorphism but it is not natural. It depends on the ordered basis we chose for L. Nevertheless, the following holds for any such choice.

Corollary 2.7. *The map $\varphi\colon U(L) \to P(L)$ constructed above is isomorphism of vector spaces with their increasing filtrations. By transport of structure determines an associative algebra multiplication on $P(L)$ which sends the product of a polynomial of degree k times a polynomial of degree ℓ to a polynomial of degree $k+\ell$ whose leading term is the product of the leading terms of the factors.*

Remark 2.8. This result should not be surprising: We saw that if L is abelian the $U(L) \cong P(L)$ as graded algebras. Also, the identity $xy - yx = [x, y]$ is a correction of the commutative relation $xy - yx = 0$ by a term of lower order. Thus, it is not too surprising that these two relations induce isomorphic graded algebras. Still it is not completely formal since one makes essential use of the Jacobi identity in the proof.

2.4 The Bi-Algebra Structure on $U(L)$ and Its Primitive Elements

Now we introduce the algebraic structure that allows us to tell when an element in $U(L)$ lies in L. Not only is $U(L)$ an associative algebra, but, as we shall see, it also has a natural co-multiplication.

Since $U(L)$ is an associative algebra with unit so is $U(L) \otimes U(L)$ when the tensor product is given the multiplication $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$. We define a map $c\colon L \to U(L) \otimes U(L)$ by setting $c(x) = x \otimes 1 + 1 \otimes x$.

Proposition 2.9. *c extends uniquely to an algebra map $c\colon U(L) \to U(L) \otimes U(L)$. The map c is an algebra homomorphism that is co-associative and co-commutative and has a co-unit. These properties are summarized by saying that $U(L)$ with this co-multiplication is a* co-associative, co-commutative bi-algebra with a co-unit.

Proof. Direct computation shows that $c(x)c(y) - c(y)c(x) = c([x, y])$ for all $x, y \in L$. Thus, $c\colon L \to U(L) \otimes U(L)$ is a map of Lie algebras from L to the $AB - BA$ Lie algebra structure on $U(L) \otimes U(L)$. Thus, c extends to an algebra map $c\colon U(L) \to U(L) \otimes U(L)$. Since $c(x)$ is symmetric under interchange of factors for all $x \in L$, the image of c is symmetric under this interchange. This is the definition of a *co-commutative* co-multiplication. Similarly, for all $x \in L$ we have

$$(1 \otimes c) \circ c(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = (c \otimes 1) \circ c(x),$$

from which it follows that $(1 \otimes c) \circ c = (c \otimes 1) \circ c$ on all elements of L. Since L generates $U(L)$ as an algebra and c is an algebra map, it follows that this equation holds for all $u \in U(L)$, which is the definition of a *co-associative* co-multiplication. Finally, the *co-unit* of c is the map $U(L) \to K$ of unital algebras that sends $x \in L$ to zero for all $x \in L$. □

Definition 2.10. An element $x \in U(L)$ is *primitive* if $c(x) = x \otimes 1 + 1 \otimes x$

Lemma 2.11. *The primitive elements form a real vector subspace of $U(L)$ containing $L \subset U(L)$.*

Proof. Exercise. □

We define the *standard co-multiplication* c_0 on the polynomial algebra $P(V)$. It is characterized by $c_0(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$ and c_0 is a homomorphism of associative, commutative algebras. It is a homework problem to show the following:

Claim 2.12. *In the polynomial algebra $P(V)$ (over a field of characteristic zero) the only primitive elements for the standard co-multiplication are the elements on V.*

There is an analogous proposition for $U(L)$.

Proposition 2.13. *The primitive elements in $U(L)$ for the co-multiplication are exactly the elements in L.*

Proof. We define an increasing filtration $F_n[U(L)\otimes U(L)] = \sum_{i+j\leq n} F_i(U(L))\otimes F_j(U(L))$. Then $c\colon U(L) \to U(L) \otimes U(L)$ preserves the filtration and hence induces a co-multiplication $c' = Gr^F(c)$ on $Gr^F_*(U(L))$, which is a homomorphism of algebras with every element in degree 1 being primitive. Thus, under the identification of $Gr^F_* U(L)$ with $P(L)$ the co-multiplication c' becomes the standard co-multiplication c_0 on polynomials.

Suppose that $a \in U(L)$ is primitive and non-zero. Since no multiple of the identity is primitive, there is $n \geq 1$ such that $a \in F_n(U(L))$ and has non-trivial projection to $F_n(U(L))/F_{n-1}(U(L))$. We shall show that $n = 1$. Let $\overline{a} \in F_n(U(L))/F_{n-1}(U(L))$ be the image of a. Since $\overline{a} \in Gr^F_*(U(L))$ is primitive, under the identification of $Gr^F_*(U(L))$ with $P(L)$, the element $\overline{a}$ is identified with a primitive element for c_0. It follows from the previous claim that $\overline{a} = 0$ unless $n = 1$. But by construction $\overline{a} \neq 0$. This implies that $n = 1$. Thus, a is the sum of an element in L and a multiple of the identity: $a = x + \lambda 1$ where $x \in L$ and $\lambda \in K$. But $c(x + \lambda 1) = x \otimes 1 + 1 \otimes x + \lambda 1 \otimes 1$, so that this element is primitive if and only if $\lambda = 0$ and consequently, if and only if $a \in L$. □