Lie Groups: Fall, 2025
Lecture III: Lie's Theorems

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1 Lie sub algebras to Lie subgroups: The Statements

The first result is that a sub Lie algebra of the Lie algebra of a Lie group G integrates to a unique Lie group (up to isomorphism) with an one-to-one immersion into G.

Theorem 1.1. *Let G be a Lie group and $\mathfrak{h} \subset \mathfrak{g}$ a sub Lie algebra. Then there is a connected Lie group H, a Lie group map $H \to G$ that is a one-to-one immersion whose differential at the identity identifies the Lie algebra of H with $\mathfrak{h}$. The image of H is G is the subgroup generated by the restriction of the exponential map to $\mathfrak{h}$.*

Theorem 1.2. *Let G_1 and G_2 be connected Lie groups with G_1 simply connected, and let $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Then there is a unique homomorphism $\psi\colon G_1 \to G_2$ with $d_e\psi = \varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$.*

Theorem 1.3. *Let G_1 and G_2 be simply connected Lie groups. Suppose that $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism then there is a unique Lie group isomorphism $G_1 \to G_2$ that induces φ on their Lie algebras.*

Proof. (Theorem 1.2 impies Theorem 1.3) Applying Theorem 1.2, there is a map of Lie groups $\psi\colon G_1 \to G_2$ whose differential at the identity is φ. In particular ψ is local diffeomorphism at the identity. This implies that the kernel of ψ is a discrete normal subgroup K and ψ factors to give an injective Lie group map $\overline{\psi}\colon G_1/K \to G_2$ that is onto a neighborhood of the identity in G_2. It follows immediately that he image of $\overline{\psi}$ is an open subset. If x is in the closure of the image of $\overline{\psi}$, then there is a sequence $g_n \in G_1/K$ with $\overline{\psi}(g_n) \mapsto x$. Hence $x\overline{\psi}(g_n)^{-1}$ converges to $e \in G_1$. Thus, for all n sufficiently large, $x\overline{\psi}(g_n^{-1}) \in \mathrm{Im}(\overline{\psi})$ and consequently, so is x. Thus, the image of $\overline{\psi}$ is also closed. Since G_2 is connected, it follows that $\overline{\psi}$ is onto and hence a bijection. The map $\overline{\psi}$ is a diffeomorphism since a bijective local diffeomorphism is a diffeomorphism. Since $|bar\psi$ is also a group homomorphsm it is a Lie group isomorphism. The last thing to note is that $\pi_1(G_1/K) \cong K$ and $\pi_1(G_2) = \{e\}$. Since $\overline{\psi}\colon G_1/K \to G_2$ is a diffeomorphism, this implies that $K = \{1\}$. This proves the existence of a map of Lie groups as required.

We turn to uniqueness. If $\rho\colon \mathbb{R} \to G_1$ is a one-parameter subgroup tangent to $X \in \mathfrak{g}_1$, then $\psi \circ \rho$ is the one-parameter subgroup in G_2 tangent to $\varphi(X)$. Thus, ψ is determined by φ on the image of the exponential map of G_1. This image generates G_1, and hence ψ is determined by φ. $\square$

2 Distributions and Foliations

Before proving Theorem 1.1 we need to discuss distributions and foliations.

Definition 2.1. A distribution of dimension k in a smooth manifold M is a smoothly varying family of tangent k-planes a $D^k(x) \subset T_xM$ for every $x \in M$. Smooth variation means that in a neighborhood U each $x \in M$ there are local vector fields $\chi_1, \cdots, \chi_k$ such that for each $y \in U$ the $\chi_i(y)$ are contained in $D^k(y)$ and are linearly independent implying that they generate $D^k(y)$. An *integral submanifold* for a distribution is a k-dimensional submanifold $P \subset M$ such that $T_pP = D^k(p)$ for every $p \in P$. Not every distribution has integral submanifolds. There is an obvious necessary condition. Namely, the distribution must be what is called *involutive*.

Definition 2.2. A distribution D^k is *involutive* if for every pair of vector fields ξ, ζ tangent to the distribution meaning that $\xi(x), \zeta(x)$ are contained. in $D^k(x)$ for every x, the Lie bracket $[\xi, \zeta]$ must be contained in D^k.

If P is an integral submanifold and ξ and ζ are vector fields tangent to P, then since Lie bracket of vector fields is natural under smooth maps, it follows that $[\xi, \zeta]$ is also tangent to P. Thus, for D^k to have integral submanifolds through each point, the distribution must be involutive. A theorem of Frobenius states the converse.

Theorem 2.3. *(Frobenius) If $\mathcal{D}^k$ is a k-dimensional involution, then near each point point there is a coordinate system $(x^1, \ldots, x^k, y^1, \ldots y^{n-k})$ such that the distribution in this neighborhood at each point is the tangent space spanned by $\{\partial/\partial x^1, \ldots \partial/\partial x^k\}$. Thus, a distribution $\mathcal{D}$ in M has a (local)* *integral submanifold through every point of $p \in M$ if and only if it is involutive; i.e., if and only if the space of vector fields tangent to $\mathcal{D}$ is a Lie subalgebra of the space of all vector fields on M. In this case any two integral submanifolds through x coincide in a neighborhood of x.*

Theorems 2.3 is. not a deep theorem. The problems associated to this lecture lead you through the proof of this result (with some hints along the way).

2.1 Flow Boxes and Foliations

Definition 2.4. (Flow Box Condition) A k-dimensional *flow box* for M is a smooth embedding of a manifold $F = U^k \times V^\ell$ where U is an open subset of $\mathbb{R}^k$ and V is an open subset of $\mathbb{R}^\ell$ onto an open subset of M. The subspaces of M that are images of subspace of $U \times V$ of he form $U \times \{v\}$ are the *horizontal subspaces* or *local leaves* of the flow box.

The *local distribution* determined by a flow box $U \times V \cong F \subset M$ is the image of the distribution on $U \times V$ that is tangent to the horizontal subspaces.

Two flow boxes are *compatible* if their horizontal distributions agree on the intersection.

Suppose that the flow box $F = U \times V$ has coordinates $(\overline{u}, \overline{v})$ and the flow box $F' = X \times Y$. By this we mean $\overline{u}$ is the coordinates for U, etc. Then the condition that F and F' are compatible is that $\partial\overline{y}/\partial\overline{u} = 0$ at all points of the intersection; i.e., that the Jacobian matrix of the overlap function be block upper triangular.

One can formulate this geometrically as well: For each a local leaf $U \times \{v\}$ of F, each connected component of $U \times \{v\} \cap F'$ lies in a local leaf of F' and this component is an open subset of that local leaf.

Definition 2.5. An *atlas* of k-dimensional flow boxes for M is a covering of M by a family of compatible k-dimensional flow boxes. A *k-dimensional foliation* of M is a covering of M by a maximal family of compatible flow boxes. (Maximal in the sense that any flow box compatible with every flow box in the collection is already in the collection.)

Given a k-dimensional foliation of M at each point $x \in M$ there is a well-defined notion of a horizontal k-plane in T_xM. This collection of horizontal is a k-dimensional distribution called the *horizontal distribution* of the foliation..

Lemma 2.6. *1. An atlas of k-dimensional flow boxes for M determines a k-dimensional foliation on M.*

2. The horizontal distribution of a foliation of is an involutive distribution.

3. An involutive k-dimensional distribution determines a unique k-dimensional foliation of which it is the horizontal distribution.

The first item is a simple application of Zorn's lemma. The second and third are direct and left to the reader.

Definition 2.7. Let $\mathcal{F}$ be a k-dimensional foliation of M determined by an atlas of flow boxes $\{F_\alpha\}_\alpha$. We introduce an equivalence relation on the points of M. There is an elementary equivalence of x to y if there is a flow box containing both of them and there is a path in a local leaf of the flow box connecting them. (Or we could require that the horizontal spaces U of the flow boxes be connected and simply require that x and y lie on the same local leaf of a flow box.) This relation is symmetric so it generates an equivalence relation. Each equivalence class is a (global) *leaf* of the foliation.

It is easier to work with connected flow boxes. This is no restriction since the connected component of a flow box inherits a flow box structure so that each flow box is a union of connected flow boxes.

Lemma 2.8. *Fix a atlas of connected flow boxes for a foliation. A leaf is a union of local leaves of flow boxes in the atlas. If local leaves A and A' from two different flow boxes are contained in a leaf and intersect each other, then their intersection is an open subset of each and the overlap function from $A \subset F$ to $A' \subset F'$ is a diffeomorphhism.*

Proof. By the definition of an equivalence relation if a leaf contains a point (u, v) if a flow box $F = U \times V$ then it contains $U \times \{v\}$. This proves the first statement in the lemma. If the local leaves $U \times \{v\}$ and $X \times \{y\}$ from different flow boxes meet then we have seen that for every point of the intersection there is a neighborhood in the intersection that is an open subset in both $U \times \{v\}$ and $X \times \{y\}$. Since $U \times \{x\}$ and $X \times \{y\}$ are smooth submanifolds of M, the overlap function between them is smooth. □

2.2 The Leaf Topology

The local leaf topology on a flow box $U \times V$ is the product of the usual topology on U and the discrete topology on V. This topology makes the flow box a (non-second countable) Hausdorff k-dimensional manifold.

By what we have seen above, if F and F' are compatible flow boxes then the leaf topolgies on F and F' are compatible in the sense that they give the same topology on $F \cap F'$. Thus, the local leaf topologies on an atlas determine a new topology, the *local leaf topology*, on M. In this topology M is a smooth k-dimensional manifold.

Proposition 2.9. *A leaf of the foliation is a connected component of the local leaf topology on M.*

Proof. Again it suffices to work with an atlas of connected flow boxes. Suppose that $\mathcal{L} = A \coprod B$ with A and B open and $A \neq \emptyset$. Then since each local leaf is connected each local leaf in $\mathcal{L}$ is contained in either A or B. Fix $x \in A$ and suppose that $y \in \mathcal{L}$. Then there is a chain $x = x_1, \ldots, x_n = y$ such that x_i and x_{i+1} and a chain of local leaves in flow boxes $U_1, \ldots, U_{n-1}$ with $x_i, x_{i+1} \in U_i$ for all $1 \leq i \leq n-1$. Then inductively we see that all the U_i are in A and hence $y \in A$. Thus, the entire local leaf is contained in A, and $B = \emptyset$. This proves that the restriction of the leaf topology to $\mathcal{L}$ makes it a connected space.

Conversely, suppose that there is a connected subspace of the leaf topology on M that contains $\mathcal{L}$ and a point x not in $\mathcal{L}$. Since the leaf topology on M is locally path connected, there is path in the leaf topology from a point of $y \in \mathcal{L}$. Since the path components of local leaf topology are the leaves, any path from y to x is contained in a finite union of local leafs. This implies that $y \sim x$ and hence $x \in \mathcal{L}$, which is a contradiction, showing that $\mathcal{L}$ is a connected component of the local leaf topology on M $\square$

Definition 2.10. The local leaf topology on M restricts to $\mathcal{L}$ to give the leaf topology on $\mathcal{L}$

Proposition 2.11. *A global leaf with the leaf topology is a Hausdorff smooth k-dimensional manifold that is smoothly one-to-one immersed in M with its usual topoogy.*

Proof. Let $\mathcal{L}$ be a leaf of the foliation. Let F be a flow box with connected horizontal space. Consider all the local leaves of F that are contained in $|mathcalL$. Each of these gives a topological embedding of a smooth manifold into $\mathcal{L}$. As we vary over all flow boxes with connected horizontal spaces we get a covering of $\mathcal{L}$ by local smooth k-dimensional manifolds, whose differential structures are compatible on the overlaps. This determines the smooth structure on $\mathcal{L}$. The inclusion of $\mathcal{L} \to M$ is a smooth immersion from this differential structure on $\mathcal{L}$ to M with its usual differential structure. Since set underlying $\mathcal{L}$ is a subset of M and the smooth map is the inclusion of the subset into M, this immersion is one-to-one. $\square$

We have not shown that the leaves are 2^{nd} countable. Assuming that M is 2^{nd} countable, this is true but requires a slightly intricate argument. I will not discuss this. In the applications to Lie groups, the second countability is automatic since connected Lie groups are 2^{nd} countable.

We have shown:

Corollary 2.12. *Let $\mathcal{D}^k$ be a k-dimensional distribution in M^n that is involutive. Then there is a foliation all of whose leaves are maximal integral submaifolds of $\mathcal{D}$.*

3 Proof of Theorem 1.1

Proof. (of Theorem 1.1) Now we apply the theory of foliations and involutive distributions to the case of a sub Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the Lie algebra of a group G. The distribution we take is the left invariant distribution whose value at e is $\mathfrak{h} \subset \mathfrak{g}$. Let us denoted it by $\mathcal{D}(\mathfrak{h})$

Take a basis for $\{X_1, \ldots, X_k\}$ for $\mathfrak{h}$. They generate left invariant vector fields $\xi_1, \ldots, \xi_k$ that are a basis at each point for $\mathcal{D}(\mathfrak{h})$ at every point. The bracket of $[\xi_i, \xi_j]$ is a left invariant vector field, and its value at the origin is $[X_i, X_j]$. Since $X_i, X_j \in \mathfrak{h}$ and $\mathfrak{h}$ is a sub Lie algebra, $[X_i.X_j] \in \mathfrak{h}$ and hence $[\xi_i, \xi_j]$ is tangent to $\mathcal{D}(\mathfrak{h})$. The general vector fields tangent to this distribution are of the form $\sum_i f_i\xi_i$ for some smooth functions $f_1, \ldots, f_k$. Then

$$[\sum_i f_i\xi_i, \sum_j g_j\xi_k j] = \sum_{i,j} f_i\xi_i(g_j)\xi_j - g_j\xi_j(f_i)\xi_i + f_ig_j[\xi_i, \xi_j].$$

The first two terms are visibly in $\mathcal{D}(\mathfrak{h})$, being functions times the ξ_i and the last term is in $\mathcal{D}(\mathfrak{h})$ by what we just showed above.

Thus, according to Theorem 2.3, the distribution $\mathcal{D}(\mathfrak{h})$ integrates to a foliation. Let H be the (global) leaf of this foliation containing the origin. With its leaf topology, H is a k-dimensional manifold smoothly one-one immersed in G. Its tangent space at the origin in $\mathfrak{h}$.

Since the foliation is invariant under left multiplication, for any $g \in G$, $g \cdot H$ is a leaf of the foliation. In particular $g \cdot H \cap H \neq \emptyset$ if and only if $g \cdot H = H$.

Claim 3.1. *$g \cdot H = H$ if and only if $g \in H$. Also, H is a subgroup of G.*

Proof. If $g \cdot H = H$ then $g = g \cdot e \in H$. Conversely, if $g \in H$, then $g \cdot e \in H$ so that $g \cdot H \cap H \neq \emptyset$ and hence, $g \cdot H = H$. This shows that H is closed under multiplication. On the other hand if $g \in H$ then $g \cdot H = H$, and hence $H = g^{-1} \cdot H$ so that $g^{-1} \in H$, showing that H is closed under taking inverses. $\square$

The inclusion $H \to G$, viewed as a map from H with the leaf topology to G with its usual topology, is a smooth map and a group homomorphism. Thus, it is a morphism of Lie groups.

The uniqueness statement follows from the uniqueness in Frobenius's theorem. $\square$

4 Proof of Theorem 1.2

Let G_1 and G_2 be groups with G_1 simply connected, and let $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. Consider the product Lie group $G_1 \times G_2$. Its Lie algebra is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the direct sum bracket. The graph of φ is a linear subspace $V \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ whose projection onto the first factor is a linear isomorphism. Since φ is a Lie algebra homomorphism $V \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Lie subalgebra.

According to Theorem 1.1 the left-invariant distribution $\mathcal{D}(V)$ on $G_1 \times G_2$ is tangent to a foliation. Furthermore, letting H be the leaf of that foliation through e with the leaf topology, the inclusion map of $i\colon H \subset G_1 \times G_2$, viewed as a map from H with the leaf topology to $G_1 \times G_2$, is a smooth on-to-one immersion of Lie groups. Since the differential of the projection $G_1 \times G_2 \to G_1$ sends V isomorphically onto T_eG_1, by equivariance, the projection $G_1 \times G_2 \to G_1$ maps the tangent plane to H at each point onto the tangent pane to G_1. That is to say the composition of i followed by the projection to G_1 is a local diffeomorphism $\rho\colon H \to G_1$.

Since G_1 is connected, it is generated by an open neighborhood of the identity. This implies that the local diffeomorphism $\rho\colon H \to G_1$ is surjective. Similarly, the kernel of ρ is discrete. Since G_1 is simply connected and H is connected, the kernel of ρ is trivial. That is to say $\rho = \pi_1 \circ i\colon H \to G_1$ is an isomorphism of Lie groups. Hence, the composite, $\psi = \pi_2 \circ \rho^{-1}\colon G_1 \to G_2$, is a Lie group homomorphism whose graph is $i(H) \subset G_1 \times G_2$. This implies that the graph of $d_e(\psi)$ in $\mathfrak{g}_1 \times \mathfrak{g}_2$ is V which, recall, is the graph of $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. It follows that $d_e(\psi) = \varphi$.

This completes the proof of Theorem 1.2.

5 Isogeny of Lie Groups

We begin by discussing covering groups of connected Lie groups.

Definition 5.1. A map of connected Lie groups $\varphi\colon G_1 \to G_2$ is a *covering Lie group* if the map is a covering projection and a homomorphism of Lie groups.

As an example, suppose that G is a connected Lie group and $K \subset G$ is a discrete, normal subgroup. Then $G \to G/K$ is a covering group.

Lemma 5.2. *A morphism of connected Lie groups $\varphi\colon G_1 \to G_2$ is a covering Lie group if and only if it induces an isomorphism of Lie algebras.*

Proof. If φ is a covering Lie group, then it is a local diffeomorphism and $d_e\varphi$ is a linear isomorphism. Since φ is a Lie group map, $d_e\varphi$ is a homomorphism of Lie algebras and hence an isomorphism of Lie algebras.

Conversely, if $d_e\varphi$ is an isomorphism, then φ is a local diffeomorphism at the identity. That is to say, there is a neighborhood U_1 of the identity in G_1 such that $\varphi|_{U_1}$ is a diffeomorphism onto an open subset U_2 of the identity in G_2. Since G_2 is connected it is generated by U_2. Thus, φ is onto.

Let $K \subset G_1$ be the kernel of φ. It is a normal subgroup. Since $\varphi|_{U_1}$ is injective, $K \cap U_1 = \{e\}$. Let $W \subset U_1$ be a smaller open neighborhood of the identity with the property that $W = W^{-1}$ and $W^2 \subset U$. We claim that $kW \cap k'W = \emptyset$ for all $k \neq k'$ elements of K. For if $w_0 \in kW \cap k'W$ then we have $kw = k'w'$ for some $w, w' \in W$. This implies that $k^{-1}k' = w(w')^{-1} \in W^2 \subset U$. Since $k^{-1}k' \in K$ and $K \cap U = \{e\}$, it follows that $k = k'$. This shows that K is a discrete subgroup of G_1.

Since $K \subset G_1$ is a normal group we have a Lie group homomorphism $\pi\colon G_1 \to G_1/K$. The above argument shows that π evenly covers the image, $\overline{W} \subset G_1/K$, of W. Clearly, $\overline{W}$ is an open neighborhood of the identity in G_1/K Now consider $\overline{g} \in G_1/K$. Left translation by $\overline{g}$ maps the image of $\overline{W}$ isomorphically to $\overline{g}\overline{W}$ a neighborhood of $\overline{g}$ in G_1/K. Multiplication by a lift $g \in G_1$ of $\overline{g}$ and maps $\pi^{-1}(\overline{W})$ isomorphically to $\pi^{-1}(\overline{g}\overline{W}) = \coprod_{k\in K} gkW$. This shows that φ is also a covering map on $\overline{g}\overline{W}$ for every $\overline{g} \in G_1/K$. Thus, $G_1 \to G_1/K$ is a covering projection.

Since $K = \ker(\varphi)$, the map φ factors through $\pi\colon G_1 \to G_1/K$ to give a map $\overline{\varphi}\colon G_1/K \to G_2$. This map is a group homomorphism and smooth so it is a map of Lie groups with trivial kernel. Since G_2 is connected, it is generated by the neighborhood $\overline{\varphi}(\overline{W})$ of the identity. Hence $\overline{\varphi}$ is surjective. Thus, $\overline{\varphi}$ is a bijective, local diffeomorphsim and hence a diffeomorphism. It is also a homomorphism of Lie groups, and hence an isomorphism of Lie groups. Hence, $\varphi\colon G_1 \to G_2$ is also a covering Lie group. $\square$

5.1 The Universal Covering Group

Proposition 5.3. *Let G be a connected Lie group and let $\widetilde{G}$ be the universal covering of G and fix $\widetilde{e} \in \widetilde{G}$ a point above $e \in G$. Then there is a unique Lie group structure on $\widetilde{G}$ with the properties that (i) $\widetilde{e}$ is the identity element and (ii) the projection $\widetilde{G} \to G$ is a Lie group homomorphism. The kernel of this homomorphism is a discrete subgroup $K \subset \widetilde{G}$ and the covering projection induces a Lie group isomorphism $\widetilde{G}/K \to G$. In particular, the Lie algebras of $\widetilde{G}$ and G are canonically identified.*

Proof. Given $g_1, g_2 \in \widetilde{G}$, let $\omega_1(t)$ and $\omega_2(t)$ be paths defined on $[0,1]$ in $\widetilde{G}$, each beginning at $\widetilde{e}$ with $\omega_i(1) = g_i$. Let $\overline{\omega}_1(t)$ and $\overline{\omega}_2(t)$ be the images of these paths in G, and let $\overline{\mu}(t) = \overline{\omega}_1(t)\overline{\omega}_2(t)$. This is a path beginning at e. Using unique path lifting, lift $\overline{\mu}$ to a path $\mu(t)$ beginning at $\widetilde{e}$. We define $g_1g_2 = \mu(1)$.

A standard argument with covering spaces shows that if we choose different paths $\omega_1'(t)$ and $\omega_2'(t)$ from $\widetilde{e}$ to g_1 and g_2, respectively, the two definitions of g_1g_2 agree. [Show that as we vary the paths, the notion of g_1g_2 is locally constant. Since $\widetilde{G}$ is simply connected two pairs of paths from $\widetilde{e}$ to g_1 and g_2 came be joined by a connected family of such pairs of paths. This and the local constancy of the resulting product, show that the product g_1g_2 is well defined.] It is direct to see that $\widetilde{e}$ acts as a two-sided identity for this multiplication and that this multiplication is associative.

Given $g \in \widetilde{G}$, one defines g^{-1} by choosing a path ω from $\widetilde{e}$ to g, projecting ω to a path $\overline{\omega}$ in G, forming the path $\overline{\omega}^{-1}(t) = (\overline{\omega}(t))^{-1}$ and lifting $\overline{\omega}^{-1}$ to a path μ beginning at $\widetilde{e}$. We define $g^{-1} = \mu(1)$. It is clear from the definitions that $gg^{-1} = g^{-1}g = \widetilde{e}$. Thus, we have defined a group structure on $\widetilde{G}$ with $\widetilde{e}$ as the identity element. Clearly, the projection mapping is a homomorphism of groups

One defines the smooth structure on $\widetilde{G}$ by requiring the projection map to be a local diffeomorphism. One checks easily group multiplication and inverse are smooth mappings in this smooth structure. Thus, the projection is a smooth map and a group homomorphism; that is to say the projection is morphism of Lie groups. □

5.2 All Covering Groups

Lemma 5.4. *Any discrete, normal subgroup of a connected Lie group is abelian and central.*

Proof. Let G be a connected Lie group and $K \subset G$ a discrete normal subgroup. Since K is normal, $gKg^{-1} = K$ for all $g \in G$. That is to say conjugation by G induces a map $G \to \mathrm{Auto}(K)$. But since K is discrete, so is $\mathrm{Auto}(K)$. But G is connected, so any map $G \to \mathrm{Auto}(K)$ is constant, meaning that the adjoint action of G on K is trivial. Thus, K is contained in the center of G and *a fortiori* is abelian. □

Corollary 5.5. *Let G be a connected Lie group and M a connected manifold. Suppose that $\pi\colon M \to G$ is a covering projection. Then there is a Lie group structure on M such that $\pi\colon M \to G$ is a covering Lie group.*

Proof. Every connected covering of G corresponds to a subgroup of $\pi_1(G, e)$. The universal covering Lie group $\widetilde{G} \to G$ corresponds to the trivial subgroup The kernel of the projection mapping $\widetilde{G} \to G$ is a discrete normal subgroup K of $\widetilde{G}$ isomorphic to $\pi_1(G)$. By the previous lemma K is central in $\widetilde{G}$.

All other connected covering spaces of G are isomorphic to $\widetilde{G}/K'$ where K' is a subgroup of K. Since K is central, K' is also central, and *a fortiori* is a normal subgroup. Thus, $\widetilde{G}/K$ inherits the structure of a Lie group from $\widetilde{G}$. Clearly, then he projection $G/K \to G$ is a covering Lie group. □

Definition 5.6. Two connected Lie groups G_1 and G_2 are *isogenous* if there is a Lie group G and Lie group maps $\varphi_i\colon G \to G_i$, for $i = 1, 2$, that are covering Lie groups.

Corollary 5.7. *Let G_1 and G_2 be connected Lie groups. Then the following are equivalent:*

- *G_1 and G_2 are isogenous.*
- *The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.*
- *The universal covering groups of G_1 and G_2 are isomorphic as Lie groups.*

Proof. If G is a covering group of both G_1 and G_2, then the universal covering group of G is also the universal covering group of G_1 and G_2. This shows the first item implies the third. The third obviously implies the first and second. The second implies the third by Theorem 1.2. □

6 Ado's Theorem

To complete the picture of the general theory of Lie groups we need a nontrivial result from the theory of Lie algebras.

Theorem 6.1. *(Ado's Theorem) Every finite dimensional real Lie algebra has a faithful finite dimension linear representation*

The proof of this theorem requires a detour through some of the more detailed parts of general Lie algebra theory. I will not prove it in this course. Nevertheless, I will use the following consequence.

6.1 Consequences of Ado's Theorem

Theorem 6.2. *Let G be a connected Lie group. Then there is an isogenous Lie group G' that admits a faithful finite dimensional representation; i.e., for some n there is a Lie group homomorphism $G' \to GL(n, \mathbb{R})$ that is a one-one immersion.*

Proof. (Assumping Ado's Theorem) By Ado's theorem, there is $n > 0$ and an embedding $\iota\colon \mathfrak{g} \subset \mathfrak{g}l(n, \mathbb{R})$ of Lie algebras. By Theorem 1.2 there is a group G' and a Lie group homomorphism $\psi\colon G' \to GL(n, \mathbb{R})$ that is a one-one immersion and whose differential $d_e\psi\colon \mathfrak{g}' \to \mathfrak{g}l(n, \mathbb{R})$ maps $\mathfrak{g}'$ isomorphically onto $\iota(\mathfrak{g})$. Since G and G' have the isomorphic Lie algebras by Corollary 5.7 they are isogenous. $\square$

Remark 6.3. It is not true that every Lie group has a faithful finite dimensional representation. In fact $\pi_1(SL(2, \mathbb{R})) \cong \mathbb{Z}$ and the universal covering group of $\widetilde{SL(2, \mathbb{R})}$ does not have a faithful finite dimensional representation.

Theorem 6.4. *Every finite dimensional Lie algebra is (up to isomorphism) the Lie algebra of a group, indeed of a simply connected group.*

Proof. (Assuming Ado's Theorem) Let L be a finite dimensional real Lie algebra. Then according to Ado's Theorem, there is an embedding $L \subset \mathfrak{g}l(n, \mathbb{R})$ for some n. Applying Theorem 1.1 there is a Lie group H and a one-one immersion $H \to GL(n, \mathbb{R})$ so that L is the Lie algebra of H. The universal covering group of H is a simply connected Lie group with Lie algebra L. $\square$