Lie Groups: Fall, 2025
Lecture I

September 9, 2025

This section starts with the basic definitions of the course – real and complex Lie groups and then gives a series of basic examples. We then study two important technical results that are used repeatedly in the subject: the existence of left-invariant metrics on a Lie group and the smooth manifold structure of the space of cosets of a sub-Lie group H of G and the quotient map $G \to H\backslash G$.

1 The Basic Definitions

Definition 1.1. A *Lie group* is a smooth finite dimensional manifold G with two structure maps, which are required to be smooth maps, $m\colon G \times G \to G$ and $\iota\colon G \to G$, together with an element $e \in G$. These structure maps define a group structure on G with m as a product, e as the identity element, and ι as the map $g \mapsto g^{-1}$. A *map* $\varphi\colon G \to H$ of Lie groups is a smooth map from the manifold underlying G to that underlying H that is also a group homomorphism

If G is a complex manifold and the structure maps m and ι are holomorphic, then G is a *complex Lie Group*. A *map* between complex Lie groups is a holomorphic map that is a group homomorhism.

Obviously, these definitions define the category of Lie groups and a category of complex Lie groups.

There is one technical issue in the definition of Lie groups and complex Lie groups; namely what we mean by a manifold. There are two conditions that are optional in the definition of a manifold: Hausdorff and 2^{nd} countable (which means that there is a countable basis for the topology). Usually, manifolds are assumed to be Hausdorff and second countable. We shall always require that the manifolds underlying Lie groups be Hausdorff.

Normally, we shall implicitly assume that they are second countable as well, but it is not essential as the following lemma shows.

Lemma 1.2. *Let G be a connected Lie group. Then G is second countable.*

Proof. We shall see later in this lecture that G is metrizable. It is a general theorem in topology that a metrizable space is second countable if and only if it has a countable dense subset. [Homework problem.]

Since G is a manifold, there is a neighborhood U of e in G diffeomorphic to an open subset of $\mathbb{R}^n$ for some $n < \infty$. We take U to be invariant under $g \mapsto g^{-1}$. The points of U with rational coordinate values form a dense open subset $Q \subset U$. Consider now $\prod_{i=1}^k U \to G$ given by $(g_1, \ldots, g_k) \mapsto g_1 \cdots g_k$. The domain of this map is an open subset of a Euclidean space and hence has a countable dense subset. The same is true of its image $U_k \subset G$. It follows that $U_\infty = \cup_{i=1}^\infty U_i$ also has a countable dense subset. The proof is completed by the following claim.

Claim 1.3. $G = U_\infty$.

Proof. Clearly, U_∞ is a non-empty, open subset of G. Since G is connected, we need only show that U_∞ is closed subset. Let $z \in G$ be a limit point of U_∞. Then gU is an open subset about z and hence $gU \cap U_\infty \neq \emptyset$. Let $h \in gU \cap U_\infty$. Since $h \in U_\infty$, it follows that $h \in U_k$ for some $k < \infty$. Also, $h = gu$ for some $u \in U$ and hence $g = hu^{-1}$ with $u^{-1} \in U$. Hence, $g \in U_{k+1} \subset U_\infty$.

Since U_∞ contains all its limit points and G is a metric space, U_∞ is a closed subset of G. □

□

Corollary 1.4. *A Lie group is second countable if and only if it has at most countably many connected components.*

1.1 Submanifolds and sub Lie groups

Definition 1.5. Let M be a smooth manifold. A *smooth submanifold* is a subset $N \subset$M with the property that for each $m \in M$ there is a local coordinate system $(x^1, \ldots, x^k)$ defined on an open set U containing the point m such that $N \cap U$ is given by the subset of U where the equations $\{x^{r+1} = \cdots = x^k = 0\}$ hold. Then N inherits a unique smooth structure such that the inclusion $N \to M$ is a smooth map. Such a map is called *a smooth embedding.* [Since we are working exclusively in the smooth category we shall drop the adjective *smooth* from the terminology both for submanifolds and embedding. It is implicit.]

Notice that if N is a submanifold of M then it is a closed subset of M. There is a converse to this. Suppose that $\varphi\colon N \to M$ is an immersion (injective differential at every point) and is a one-to-one map. Then the image $\varphi(N)$ is a submanifold of M if and only if it is a closed subset. An example showing that the image is not automatically closed is given by the map

$$\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

where $f(t) = (t, \pi t)$.

Definition 1.6. Let G be a Lie group. A *Lie subgroup* $H \subset G$ is a smooth submanifold H of G that is closed under the product and inverses and contains the identity element[1].

The terminology is justified by the following lemma.

Proposition 1.7. *If $H \subset G$ is a sub-Lie group, then the restriction of the product and inverse of G to H give H the structure of a Lie group and the inclusion $H \subset G$ is a morphism of Lie groups.*

Proof. We consider the case when G is connected, and leave the generalization to non-connected groups to the reader. Since H is closed under product and inverses and contains the identity, the restriction of the group structure maps from G to H define the structure of a group on H. We need only see that the product and the inverse are smooth maps of H. But they are smooth maps of G and H is a smooth submanifold invariant under the maps. Hence, the restriction of the maps to H are smooth. This establishes that H with the induced structures is a Lie group. The inclusion $H \subset G$ is a smooth map and a group homomorphism and hence, by definition is a morphism of Lie groups. □

There is an analogue of the first part of Lemma 1.7 for one-to-one immersed subgroups.

Lemma 1.8. *Let G be a Lie group. Suppose that H is a smooth manifold and $\varphi\colon H \to G$ is a one-to one smooth immersion whose image is a subgroup of G. Then there is a unique Lie group structure on H so that φ is a homomorphism of Lie groups.*

[1]In the literature one sometimes finds the more general notion of sub Lie group where the submanifold is not required to be closed, just to be one-to-one immersed.

Proof. Since H is a smooth manifold and a group, we need only show that group multiplication and inverse are smooth maps. Let $(h, h') \in H \times H$. There there are neighborhoods U, U' and V of h, h' and hh', respectively, such that $\varphi\colon U \to G$ and $\varphi\colon U' \to G$ and $\varphi\colon V \to G$ are embeddings onto smooth (locally closed) submanifolds. Taking U and U' sufficiently small we can arrange that the product in G maps $U \times U' \to V$. Since the group multiplication of G is smooth the composition $U \times U' \to V \subset G$ is smooth, and since V is a locally closed smooth submanifold of G, this implies that $U \times U' \to V$ is smooth.

The argument for the inverse map is analogous. □

2 Examples

2.1 Real Lie Groups

Groups naturally arise as symmetry groups of some mathematical structure, so they come with their defining action. Most Lie groups, complex Lie groups, or linear algebraic groups arise in this way.

Any discrete group is a Lie group. If we require, as one often does, that a manifold must be second countable, then only the countable discrete groups are Lie groups. Of particular interest are the finite groups.

Example 1. The symmetries of a square in the plane, meaning a Euclidean isometry of the square onto itself consists of rotations through multiples of $\pi/2$ around the central point of the square, together with flips, either about a line bisecting two opposite sides or a line passing through two opposite vertices. These form a group of order 8 with a normal subgroup being the group of 4 rotations. Similarly, the Euclidean symmetries of a regular n-gon in the plane is a group of order $2n$ with a normal subgroup being the cyclic group consisting of the n rotational symmetries.

Example 2. The real line $\mathbb{R}$ with m being addition and $\iota(x) = -x$ is a Lie Group, the *additive group over* $\mathbb{R}$. The non-zero real numbers $\mathbb{R}^*$ under multiplication form a Lie group. Similarly, the additive group of complex numbers $\mathbb{C}$ and the multiplicative group of non-zero complex numbers $\mathbb{C}^*$ are complex Le groups. The unit circle in the complex plane with product being product of complex numbers and ι being inverse of complex numbers is a Lie group. Indeed, it is a real Lie subgroup of the real Lie group underlying the complex Lie group $\mathbb{C}^*$.

Example 3. Let V be a finite dimensional real vector space. Then the general linear group of V, denoted $GL(V)$, is a Lie group under matrix multiplication and matrix inverse. Let V have a volume form $\mathrm{vol}(V) \in \Lambda^{\mathrm{top}}V$. Then $SL(V)$ the subgroup of $GL(V)$ of matrices that preserve this form is a subgroup. Check that $SL(V)$ is a smooth submanifold of $GLV)$. Let $\mathbb{R}^n$ have its usual coordinates and volume form, Then $GL(\mathbb{R}^n)$ is the group of $n \times n$ matrices of real numbers with non-zero determinant and $SL(\mathbb{R}^n)$ is the subgroup of matrices of determinant one.

Example 4. Let Q be a non-degenerate quadratic form on a finite dimensional real vector space V. W define $O(Q)$, the *orthogonal group of* Q, to be the subgroup of $GL(V)$ that leaves Q invariant in the sense that $A \in GL(V)$ is in $O(Q)$ if and only if $Q(Av) = Q(v)$ for all $v \in V$. Check that $O(Q)$ is a smooth submanifold of $GL(V)$ that closed under the product and taking inverses and contains the identity. Applying the above lemma, we see that it is a sub-Lie group of $GL(V)$ and hence is a Lie Group in its own right.

The example $O(n)$ is the orthogonal group of the standard positive definite Euclidean inner product on $\mathbb{R}^n$. The group $SO(n)$ is the subgroup of $O(n)$ of matrices of determinant 1. Show that $SO(n)$ is the component of the identity of $O(n)$.

Example 5. Let V be a finite dimensional real vector space and $\omega \cdot V \otimes V \to \mathbb{R}$ be a non-degenrate skew-symmetric form. Then the subgroup of $GL(V)$ that preserves this form, $Symp(V)$ is called a *symplectic group*. Check that this is a Lie subgroup of $GL(V)$. In the special case when $V = \mathbb{R}^{2n}$ with symplectic form

$$\omega(\sum_{i=1}^{2n} s_i e^i, \sum_{i=1}^{2n} t_i e^i) = \sum_{i=-1}^{n} s_{2i-1}t_{2i} - s_{2i}t_{2i-1}$$

this is the symplectic group $Symp(2n)$.

Example 6. If G_1 and G_2 are Lie groups, then the product smooth manifold $G_1 \times G_2$ is naturally a Lie group under the product operations. Notice that $G_1 \times \{e\}$ and $\{e\} \times G_2$ are sub Lie groups of $G_1 \times G_2$, and this is a categorical product in the category of Lie Groups.

2.1.1 Some Counter-Examples

Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The translation structure on $\mathbb{R}^2$ induces an Abelian group structure on T^2 that makes it a compact Lie group. Any sub-Lie group $H \subset T^2$ is a closed subset of T^2 and hence is compact. As a result every connected sub Lie groups of T^2 is isomorphic to one of T^2, S^1,$\{e\}$. If $\mathbb{R} \subset \mathbb{R}^2$ is a line through the origin in an irrational direction, then it induces an injective map $\mathbb{R}^1 \to T^2$ map of Lie groups whose image is not compact and hence not a sub Lie group.

Notice that there is a quotient space $T^2/\mathbb{R}^1$ inherits a group structure and also is is locally isomorphic to $\mathbb{R}^1$ with local coordinates in which the group structure is smooth. But the quotient is not a Lie group since it is not Hausdorff.

There are similar examples in higher dimensional tori of all possible codimensions ≥ 1.

The examples show that in general if $\rho\colon H \to G$ is a map of Lie groups, then the image is not necessarily a sub Lie group of G.

2.2 Examples of Complex Lie groups

As in the real case, we have:

Example 7. If G is a complex Lie Group and $H \subset G$ is a complex submanifold containing the identity element of G and closed under the product operation and the inverse map, then H together with the restriction to H of these structure maps is a complex Lie Group.

Example 8. A *complex linear algebraic group* is a complex algebraic sub variety of $M(n \times n, \mathbb{C})$ contained in the Zariski open subset $GL(n, \mathbb{C})$ and closed under matrix multiplication and inverses.

Example 9. If V is a finite dimensional complex vector space then its complex linear automorphisms form a complex Lie Group. Of course, we can assume that V is isomorphic to $\mathbb{C}^n$ for some $n \geq 0$. Thus, for some $n \geq 0$ the complex Lie group $GL(V)$ is isomorphic to the complex Lie Group $GL(n, \mathbb{C})$, the group of invertible $n \times n$ complex matrices. The product is matrix multiplication and the inverse is the matrix inverse. The group is an open subset of the complex vector space $M(n \times n, \mathbb{C})$ of complex $n \times n$ matrices. In fact, being the complement of the divisor where $\{\det = 0\}$, $GL(n, \mathbb{C})$ is a Zariski open set and is a linear algebraic group over $\mathbb{C}$. We also have $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ of matrices of determinant 1 also a linear algebraic group over $\mathbb{C}$, and hence a complex Lie Group. For any non-degenerate complex quadratic form Q on $\mathbb{C}^n$ we have its complex orthogonal group, defined as in the real case. This also is a linear algebraic group over $\mathbb{C}$ and hence a complex Lie group.

Example 10. For a on-degenerate symmetric complex bilinear form Q on $\mathbb{C}^n$ we have the complex orthogonal group $G(Q)$ of linear automorphisms preserving Q. Similarly, for a non-degenerate, skew symmetric, complex bilinear form on $\mathbb{C}^n$ we have the complex symplectic group, again a linear algebraic group over $\mathbb{C}$, and hence a complex Lie group.

Exmple 11. Consider a maximal lattice $\Lambda \subset \mathbb{C}$. By definition Λ is generated by two elements that are linearly independent over $\mathbb{R}$. The quotient $\mathbb{C}/\Lambda$ is a compact complex curve diffeomorphic to $S^1 \times S^1$. Addition on $\mathbb{C}$ induces a group structure on $\mathbb{C}/\Lambda$ that makes it a complex Lie group. This is an *elliptic curve*. Show that it is not a complex linear algebraic group.

Example 12. More generally, if $\Lambda \subset \mathbb{C}^n$ is a lattice, meaning an integral basis for Λ is an $\mathbb{R}$-basis fior $\mathbb{C}^n$, then addition on $\mathbb{C}^n$ descends to a complex Lie group structure on the quotient $\mathbb{C}^n/\Lambda$. Such groups are *complex tori*.

3 Invariant metrics on G

Recall that a *Riemannian metric* g on a smooth manifold M is a smoothly varying positive definite inner product on the tangent spaces of M. In local coordinates $(x^1, \ldots, x^n)$ for M, the metric is written as $g = \sum_{i,j} g_{ij} dx^i \otimes dx^j$ where the $g_{i,j}$ are smooth functions of $(x^1, \ldots, x^n)$ and the matrix $g_{i,j}(x^1, \ldots, x^n)$ is symmetric and positive definite for every $(x^1, \ldots, x^n)$ in the coordinate patch.

Suppose that M is connected and g is a Riemannian metric.

Theorem 3.1. *Define a distance function $d\colon M \times M \to \mathbb{R}^{\geq 0}$ by*

$$d(p,q) = \inf L(\omega)$$

as ω ranges over all piecewise smooth curves from p to q and the length of a smooth curve $\gamma\colon [a,b] \to M$ is $\int_a^b \sqrt{g(\gamma'(t), \gamma'(t))}dt$ and the length of a piecewise smooth curve is the sum of the lengths of its smooth pieces. Then d is a distance function

Proof. Being the infimum of non-negative functions, $d(p,q) \geq 0$ for all $p, q \in M$. It is clearly satisfies the triangle inequality [Adjoin paths.] and is symmetric. [Reverse the path.] It is more subtle to show that $d(p,q) > 0$ for $p \neq q$.

Fix $p \in M$ and fix an orthonormal basis $\{e^1, \ldots, e^n\}$ for T_pM. Take coordinates near p so that p is the origin and $e^i = (\partial/\partial x^i)(p)$. Let $|v|^2$ denote the Euclidean norm in these coordinates. Then $g(v,v) = |v|^2$ for every tangent vector at p. There is $0 < \epsilon < 1/2$ such that for all q in the ϵ-ball about 0 in these coordinates the metric $|g_q(v,v) - |v|^2 < \epsilon|v|^2$. Thus, for any piecewise curve γ in this neighborhood the g-length of γ is at least

$(1-\epsilon)$ times the - Euclidean length of γ, and for any piecewise curve starting at p and leaving the ball has length at least $\epsilon(1-\epsilon)$. This shows that if q is in the ϵ-ball then the distance from p to q is at least $(1-\epsilon)$ times the Euclidean distance from p to q, and if q is not in this ball then the distance from p to q is at least $\epsilon(1-\epsilon)$. □

Corollary 3.2. *Let G be a Lie group. There is a Riemannian metric on G on G that is invariant under left multiplication by every element $g \in G$. The associated metric is also invariant under left multiplication by every $g \in G$ in the sense that for every $p, q, g \in G$ we have $d(gp, gq) = d(p, q)$.*

Proof. Fix a positive definite symmetric inner product $\langle \cdot, \cdot \rangle_e$ on T_eG. For each $g \in G$ and $v_1, v_2 \in T_gG$, define $\langle v_1, v_2 \rangle_g = \langle g^{-1}v_1, g^{-1}v_2 \rangle_e$. This is a family of positive definite inner products on the tangent spaces of G. Since left multiplication by g varies smoothly with g these inner products vary smoothly and hence form a Riemannian metric. By definition it is invariant under left multiplication by every $g \in G$. The associated metric is then also invariant under left multiplication by every $g \in G$. □

Definition 3.3. We call such Riemannian metrics and metrics *left-invariant*.

N.B. There is an analogous theory of right-invariant metrics.

4 The Space of Right Cosets of a Lie Subgroup

Theorem 4.1. *Let G be a Lie group and $H \subset G$ a Lie subgroup (including that H is a closed subset). Consider the (left) action of H on G by left multiplication $H \times G \to G$ given by $(h, g) \mapsto hg$. The orbit through g is Hg, which is a* right coset *of H if G. Thus, the quotient space of this action is the space of $H\backslash G$ of right cosets of H. This space has the structure of a smooth (Hausdorff) manifold in such a way that the projection $G \to H\backslash G$ is a submersion (i.e., has surjective differential at every point) and is a locally trivial fiber bundle with fibers isomorphic to H.*

Proof. Lt $\pi\colon G \to H\backslash G$ be the quotient map sending $g \mapsto Hg$. We give $H\backslash G$ the quotient topology: a subset of $U \subset H\backslash G$ is open if and only if $\pi^{-1}(U)$ is open.

Let n and k denote the dimensions of G and H, respectively. Since H is a closed subspace of G, there is diffeomorphism $\mu\colon B \to B'$ where B is the open unit ball in $\mathbb{R}^n$ centered at 0 and B' is an open neighborhood of $e \in G$

with the property that (i) $\mu(0) = e$ and (ii) setting $U = H \cap B'$, $\mu^{-1}(U)$ is the subspace $\{x^{k+1} = \cdots = x^n = 0\} \cap B$. Let

$$S = B \cap \{x^1 = \ldots = x^k = 0\} \quad \text{and} \quad S' = \mu(S) \subset G.$$

The point of intersection $H \cap S'$ is $e \in G$. Let $\psi_0 \colon U \times S \to G$ be the map $\psi_0(h, s) = h \cdot \mu(s)$. The differential of $\psi_0(e, 0)$ is an isomorphism. Thus, possibly after replacing S and U by smaller neighborhoods of 0, we can assume that the product map $\psi_0 \colon U \times S \to G$ is a diffeomorphism onto an open subset of B'.

We define $\psi \colon H \times S \to G$ by $\psi(h, s) = h\mu(s)$. This map is a local diffeomorphism $\psi \colon H \times S \to G$.

Claim 4.2. *There is a neighborhood T of 0 in S such that the restriction $\psi \colon H \times T \to G$ is a one-one map.*

Proof. If no such neighborhood T of $0 \in S$ exists, then there are sequences (h_n, s_n) and (h'_n, s'_n) in $H \times S$ with $s_n \mapsto 0$ and $s'_n \mapsto 0$ such that for all n, $(h_n, s_n) \neq (h'_n, s'_n)$ in $H \times S$ yet $h_n\mu(s_n) = h'_n\mu(s'_n)$. It follows that $\mu(s_n) = h_n^{-1}h'_n\mu(s'_n)$ and hence that $h_n^{-1}h'_n = \mu(s_n)\mu(s'_n)^{-1}$. Thus, $h_n^{-1}h'_n$ converges to 0 as $n \mapsto \infty$ and hence for all n sufficiently large $h_n^{-1}h'_n \in U$.

We have shown that for all n sufficiently large both (e, s_n) and $(h_n^{-1}h'_n, s'_n)$ lie in $U \times S$ and have the same image under ψ_0. Since ψ_0 is a diffeomorphism, this implies that for all n sufficiently large we have $(e, s_n) = (h_n^{-1}h_n, s'_n)$; i.e., for all n sufficiently large $h_n = h'_n$ and $s_n = s'_n$. This contradicts the fact that $(h_n, s_n) \neq (h'_n, s'_n)$ for all n and completes the proof of the claim. □

Fix T with the property that $\psi|_{H\times T}$ is one-to-one. Since $\psi \colon H \times T \to G$ is the restriction of a local diffeomorphism to an open subset of the domain, it is a local diffeomorphism. We have just seen that it is one-to-one. Consequently, it is a diffeomorphism onto an open subset of G. Obviously, this open subset contains H and is invariant under the left action of H. It follows easily from the definition of the quotient topology that the map $T \to H\backslash G$ given by $t \mapsto Ht$ is a homeomorphism from T onto an open subset of G containing the right coset He.

Clearly by right G-equivariance, such local coordinates exist around any orbit Hg. Suppose given two such diffeomorphisms onto open subset $\psi_0 \colon H \times T_0 \to G$ and $\psi_1 \colon H \times T_1 \to G$ where $\mu_i(T)$ is centered at $g_i \in G$. If the images of these two maps have non-empty intersection, say $\psi_0(h_0, t_0) = \psi_1(h_1, t_1)$, then for small a neighborhood V_0 of t_0 in T_0 the map $\psi_0 \colon \{h_0\} \times V_0 \to G$ is a smooth map whose image is contained in the image of ψ_1. Hence, the composition of $\{h_0\} \times V_0 \xrightarrow{\psi_0} G$ followed by ψ_1^{-1} and then projection to T_1 is a smooth map. This is exactly the transition function from the chart on T_0 given by ψ_0 and the chart on T_1 given by ψ_1 near the point Ht_0 of $H\backslash G$. This shows the transition functions are smooth and hence they define a smooth structure on the quotient space.

Clearly, in this smooth structure, the quotient map is a smooth map with surjective differential and kernel the tangent space to the orbits. Thus, the quotient map is a locally trivial smooth fiber bundle.

It remains to show that $H\backslash G$ is Hausdorff. By Corollary 3.2 we impose a left-invariant metric on G. Then we use this to define a distance function on the orbit space: $d(Hx, Hy)$ is defined to be the infinmum of the distances between $a \in Hx$ and $b \in Hy$.

Claim 4.3. *This defines a metric on $H\backslash G$.*

Proof. Clearly, $d(Hx, Hy) = d(Hy, Hx) \geq 0$ and the triangle inequality holds. We need to show that if $Hx \neq Hy$ then $d(Hx, Hy) > 0$. Suppose that Hx and Hy are distinct orbits. Since x and Hy are disjoint closed subspaces of a metric space with $\{x\}$ being compact, the distance between them is positive. On the other hand, by invariance of the distance function under left multiplication by G and *a fortiori* by H, for any $h \in H$ the distance between between hx and Hy is equal to the distance between x and Hy. Thus, the distance between Hx an Hy is positive. □

Since the quotient is a metric space, it is Hausdorff. □

Definition 4.4. We call T as above a *slice for the action of H on G at the identity.* For any $g \in G$ a *slice for the action of H on G at g* is then the image of a slice for the action at the identity under right multiplication by g.

Definition 4.5. A sub Lie group $K \subset G$ is said to be *normal* if K is a normal subgroup of G in the usual group-theoretic sense.

Lemma 4.6. *If $K \subset G$ is a normal Lie subgroup, then the space of left cosets $K\backslash G$ has the structure of a Lie group such that the projection $G \to K\backslash G$ is a homomorphism of Lie groups.*

Proof. Since K is a normal subgroup of G, the group structure on G induces a group structure on $K\backslash G$ in such a way that the projection $G \to K\backslash G$ is a group homomorphism. We have just seen that $K\backslash G$ is a smooth manifold and that the projection $G \to K\backslash G$ is a smooth map. It remains only to show that the structure maps for the group structure on $K\backslash G$ are smooth. Let us consider the multiplication map $\mu\colon K\backslash G \times K\backslash G \to K\backslash G$. Fix points $x, y \in K\backslash G$. Lift these to points $\widetilde{x}, \widetilde{y} \in G$ and let $S_{\widetilde{x}}, S_{\widetilde{y}}$ be slices from the projection mapping $G \to K\backslash G$ at $\widetilde{x}$ and $\widetilde{y}$, respectively. Let $S_{\widetilde{x}\widetilde{y}}$ be a slice for the projection $G \to K\backslash G$ at $\widetilde{x}\widetilde{y}$. Choosing $S_{\widetilde{x}}$ and $S_{\widetilde{y}}$ sufficiently small, we can assume that the image of the product $\mu(S_{\widetilde{x}} \times S_{\widetilde{y}})$ is contained in $K \times S_{\widetilde{x}\widetilde{y}} \subset G$. It is a smooth map. Thus, the composition

$$S_{\widetilde{x}} \times S_{\widetilde{y}} \xrightarrow{\mu} K \times S_{\widetilde{x}\widetilde{y}} \xrightarrow{\pi_2} S_{\widetilde{x}\widetilde{y}}$$

is also smooth. This is the restriction of the multiplication map for the quotient to $S_{\widetilde{x}} \times S_{\widetilde{y}}$.

The argument for the inverse is similar. $\square$