Lie Groups: Fall, 2022
Lecture VI
Simple Lie Algebras and of Compact Lie Groups

December 2, 2022

1 Introduction

1.1 Semi-simple matrices

Let us begin by review some of the standard factors about complex matrices, facts that will be useful later on. A complex matrix is *semi-simple* if it is diagonalizable, i.e., conjugate to a diagonal matrix. (Notice that the image under the exponential mapping of s semi-simple element is an (automatically semi-simple) action of $\mathbb{C}^*$ on $\mathbb{C}^n$.) A matrix is *nilpotent* if some positive power of it is zero. Notice both of these notions are invariant under conjugation.

The *Jordan canonical form* of a complex matrix X is a decomposition $X = X_{ss} + X_n$ where X_{ss} is semi-simple and X_n is nilpotent and X_{ss} and X_n commute. It is easy to see that this decomposition is unique.

Lemma 1.1. *For $X \in \mathfrak{g}l(n,\mathbb{C})$ both X_{ss} and X_n are given by polynomial expressions. in X, $s(X)$ and $n(X)$ respectively, where each of $s(X)$ and $n(X)$ has zero constant term.*

Proof. Let $X \in \mathrm{End}(V)$ for a finite dimensional complex vector space V. Let $p[T] \in \mathbb{C}[T]$ be the minimal polynomial for X. Then $p(X)\colon V \to V$ is the zero map, so that V is a module over $\mathbb{C}[T]/p(T)$ where T acts on V as X.

We write

$$p(T) = \prod_{\lambda_i \in \Lambda} (T - \lambda_i)^{m(\lambda_i)}.$$

Recall that $\mathbb{C}[T]$ is a p.i.d. and the ideals generated by $(T-\lambda_i)^{m(\lambda_i)}$ are relatively prime for different λ_i.

Case 1. $p(T) = T^k$ **for some** $k \geq 1$**.** In this case X is nilpotent and the polynomials in questions are $s(X) = 0$ and $n(X) = X$.

Case 2. $p(T) = (T - \lambda)^n$ **for some** $\lambda \neq 0$**.** In this case, X_{ss} is the diagonal matrix with λ down the diagonal; i.e., X_{ss} acts by multiplication by λ on V. Since $(T-\lambda)^n$ and (T) are relatively prime ideals, we can write

$$1 = TA(T) + (T - \lambda)^n B(T).$$

Since V is a module over $\mathbb{C}[T]/(T-\lambda)^n$ with T acting by X, the polynomial $XA(X)$ is multiplication by 1 on V. Hence, $\lambda X A(X) = X_{ss}$, and $X(1 - \lambda A(X)) = X_n$.

Case 3. There is more than one eigenvalue. In this case

$$p(T) = \prod_{\lambda_i \in \Lambda} (T - \lambda_i)^{m(\lambda_i)}$$

where $\#\Lambda \geq 2$. Since the ideals $(T - \lambda_i)^{m(\lambda_i)}$ are relatively prime, it follows that for each i there are polynomials $A_i(T)$ and $B_i(T)$ such that

$$1 = A_i(T)(T - \lambda_i)^{m(\lambda_i)} + B_i(T) \prod_{j \neq i} (T - \lambda_j)^{m(\lambda_j)}.$$

This proves that

$$V = \mathrm{Ker}((T - \lambda_i)^{n_i}) \oplus \mathrm{Ker}(\prod_{j \neq i} (T - \lambda_j)^{n_j}).$$

Set $P_i(T) = B_i(T) \prod_{j \neq i} (T - \lambda_j)^{m(\lambda_j)}(v)$. Then

$$P_i(X)(v) = v \quad \text{for any} \quad v \in \mathrm{Ker}((X - \lambda_i)^{m(\lambda_i)})$$

and

$$P_i(X)(v) = 0 \quad \text{for any} \quad v \in \mathrm{Ker}(\prod_{j \neq i} (X - \lambda_j)^{m(\lambda_j)}).$$

Thus $X_{ss} = \sum_i \lambda_i P_i(X)$ and $X_n = X - \sum_i \lambda_i P_i(X)$ are the polynomial expressions for X_{ss} and X_n in terms of X. Since each P_i is contained in a maximal ideal of the form $(T - \lambda_j)$, these polynomials have zero constant term, as does their sum. □

1.2 Joint Eigenspaces for commuting semi-simple elements

Suppose that we have a linear space $\mathcal{M}$ of commuting complex matrices of size $n \times n$, say generated by $M_1, \ldots, M_k$. Suppose that each M_i is semi-simple. Then any eigenspace $E_i(\alpha)$ for M_i is left invariant under M_j for all j. It follows easily from this that there is a basis $\{e_1, \ldots, e_n\}$ for $\mathbb{C}^n$ in which all the matrices of $\mathcal{M}$ are diagonal. The one-dimensional linear subspace L_i spanned by the basis element e_i is a joint eigenspace for all the matrices of $\mathcal{M}$ and its joint eigenvalue is a linear map $\lambda_i \colon \mathcal{M} \to \mathbb{C}$ giving the action of the family on L_i.

1.3 Plan for this Lecture

In this lecture we shall prove results for about presentations of $\mathfrak{sl}(n, \mathbb{C})$ and discuss how these result generalize to a broad class of complex Lie Algebras, namely semi-simple complex Lie Algebras.

Definition 1.2. A (real or complex) Lie algebra is *simple* if it of dimension > 1 and if it contains no non-trivial ideal. A Lie group is *semi-simple* if it is a direct sum of commuting simple Lie algebras. For a semi-simple Lie algebra $\mathfrak{g}$ a *Cartan* subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ each element of which maps to a semi-simple element under the adjoint representation of $\mathfrak{g}$.

Here is the basic result, which we shall establish for $\mathfrak{sl}(n, \mathbb{C})$ but which holds for any complex semi-simple Lie Algebra.

Theorem 1.3. *Let $\mathfrak{g}$ be a semi-simple complex Lie Algebra. Then any finite dimension complex representation of $\mathfrak{g}$ is completely reducible.*

Assuming this let us show that the (finite dimensional) representation theory of semi-simple Lie algebras is determined by the representation theory of their simple factors.

Proposition 1.4. *Let L be a semi-simple complex Lie algebra and $L = L_1 \oplus L_2 \oplus L_k$ a decomposition into simple factors.*

(i) Suppose that $V_1, \ldots, V_k$ are irreducible representations of $L_1, \ldots, L_k$. Form $V = \otimes^i V_i$ and define an action of L on V by

$$(X_1, \ldots, X_n) \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_i v_1 \otimes \cdots \otimes X_i(v_i) \otimes \cdots \otimes v_k.$$

This is an irreducible representation of L.

(ii) Two irreducible representations of L as in (i) are isomorphic if and only if the irreducible representations of the factors L_i out of which they are built are isomorphic.

(iii) Every irreducible representation of L is of the form given in (i).

Proof. Let's start by proving (iii). The proof is by induction on the number of simple factors. The statement is tautological if L has only one simple factor. Suppose that we know the result for semi-simple algebras with fewer that k simple factors and let $L = L_1 \oplus \cdots \oplus L_k$ be semi-simple with the L_i as its simple factors. We write $L = L_1 \oplus L'$ where L_1 is simple and L' is semi-simple with $(k-1)$ simple factors. Suppose that V is an irreducible representation of L. Consider the decomposition of V as an L_1-module into irreducible L_1 representations, grouped into isomorphism classes

$$V = W_1 \oplus \cdots \oplus \oplus W_j,$$

with each W_i being a direct sum of isomorphic L_1-modules and with the irreducible components of W_i and W_j being non-isomorphic for $i \neq j$. Since L' commutes with L_1, the action of L' preserves the L_1 structure on V and hence preserves the isomorphism types of the L_1-module factors $W_1, \ldots, W_j$. By irreducibility of the L-module this implies that there is only one isomorphism class of irreducible L_1 representation occurring in this decomposition. Thus, we have an isomorphism of L_1-modules $V = \oplus_{i \in I} W$ where W is an irreducible L_1 module.

We fix such a direct sum decomposition, and let $W_1, \ldots, W_n$ be the direct sum factors and π_j the projection of the direct sum onto the j^{th}-factor. By Shur's lemma, the composition of $W_i \subset V \xrightarrow{\pi_j} W_j$ is given by a scalar $\lambda_{i,j}$. That is to say using the direct sum decomposition to write $V = W \otimes \mathbb{C}^n$ the elements of L' are given by $\mathrm{Id}_W \otimes \alpha$ for $\alpha \in \mathfrak{gl}(n, \mathbb{C})$. This gives a representation of L' into $\mathfrak{gl}(n, \mathbb{C})$ so that the action of L on $W \otimes \mathbb{C}^n$ is given by $(X_1, X')(w \otimes v) = X_1(w) \otimes v + w \otimes X'(v)$, and an L-module map from this tensor product to V. Since V is simple and the map is non-trivial, it must be surjective. On the other hand the dimension of both V and the tensor product are the product of the dimensions of the representations of L_1 and L'. Hence the map is an isomorphism of L representations.

By induction the irreducible representation of L' is a tensor product of irreducible representations of the L_i for $2 \leq i \leq j$. This proves Item (iii).

Let us consider the first item. Again we argue by induction on the number of simple summands. When there is only one such, the result is immediate. Using the notation and assumption as in the proof of (iii), suppose W is an irreducible L_1 representation and U is an L' module that is tensor product of irreducible representations of its simple summands and the the representation of $L \otimes (W \otimes U) \to (W \otimes U)$ is the tensor product of the L_1 and L' representations. By induction U an irreducible L' representation. Then it is clear that any non-zero L_1-module of $W \otimes U$ is of the form $W \otimes U'$ where U' is a non-zero linear subspace of U'. Analogously, any non-zero L'-module of $W \otimes U$ is of the form $W' \otimes U$ where W' is a non-zero linear subspace of W. Any non-zero L-submodule must be described in both these ways, and hence is $W \otimes U$, proving that this representation is irreducible.

If $W \otimes U$ and $W' \otimes U'$ are isomorphic irreducible L-modules, then the L_1 structure is a direct sum of copies of W and the L' structure is a direct sum of copies of U'. Item (ii) follows immediately. by induction. □

2 Simple and Semi-Simple Lie Algebras

Let us see turn now to the proto-typical simple Lie algebra.

Lemma 2.1. *(i) For any $n \geq 2$ the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is a simple complex Lie algebra. A Cartan subalgera consists of the space of matrices in $\mathfrak{sl}(n, \mathbb{C})$ whose off-diagonal entries are all* 0*.*

(ii) The Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ for $n \geq 2$ is not a simple Lie algebra.

Proof. (ii): The diagonal matrices are in the center of $\mathfrak{gl}(n, \mathbb{C})$ and hence are a non-trivial ideal.

(i): Let $\mathfrak{h} \subset \mathfrak{sl}(n, \mathbb{C})$ be the abelian subalgebra of matrices (of trace 0) that have all off diagonal entries equal to zero.

Claim 2.2. *There is a basis for $\mathfrak{sl}(n, \mathbb{C})$ such that in this basis the restriction of the adjoint representation of $\mathfrak{sl}(n, \mathbb{C})$ to $\mathfrak{h}$ is diagonal*

Proof. For any $i \neq j$ with $1 \leq i, j \leq n$ let $E_{i,j}$ be the matrix in $\mathfrak{sl}(n)$ whose only non-zero entry is in the (i, j) place and is 1. Let $L_{i,j}$ be the subspace spanned by $E_{i,j}$. There is a direct sum decomposition

$$\mathfrak{sl}(n\mathbb{C}) = \mathfrak{h} \oplus_{i,j} L_{i,j}.$$

The restriction of the adjoint action of $\mathfrak{sl}(n, \mathbb{C})$ to $\mathfrak{h}$ preserves this decomposition and the action of $\mathfrak{h}$ on itself is trivial. Let $z_1, \ldots, z_n \colon \mathfrak{h} \to \mathbb{C}$ record the diagonal entries of elements in $\mathfrak{h}$. Of course, $\sum_{i=1}^n z_i = 0$. Then the adjoint action of $H \in \mathfrak{h}$ on $L_{i,j}$ is scalar multiplication by $z_i(H) - z_j(H)$. The required basis is any basis of the form a basis for $\mathfrak{h}$ union $\{E_{i,j}\}$ where, as before the indexing set is $1 \leq i, j \leq n; i \neq j$. □

The space $\mathfrak{h}$ is the eigenspace with joint eigenvalues 0 for the adjoint action of $\mathfrak{h}$, and as we have just seen $L_{i,j}$ is the one-dimensional joint eigenspace with joint eigenvalue $z_i - z_j : \mathfrak{h} \to \mathbb{C}$.

So it is clear that $\mathfrak{h}$ is an abelian subalgebra and the image of $\mathfrak{h}$ under the adjoint representation are diagonalizable. Also, it is clear from the adjoint action of $\mathfrak{h}$ on the $L_{i,j}$ that $\mathfrak{h}$ is a maximal abelian subalgebra. This proves that it is a Cartan subalgebra.

Lastly, we need to see that there are no non-trivial ideals in $\mathfrak{sl}(n,\mathbb{C})$. Let $I \neq 0$ be an ideal of $\mathfrak{sl}(n,\mathbb{C})$.

Definition 2.3. We let $H_{i,i}$ be the matrix with 1 in the (i,i) place and zero elsewhere. (Notice that $H_{i,i}$ is not an element of $\mathfrak{sl}(n,\mathbb{C})$.)

Claim 2.4. *There is a pair (i,j) with $i \neq j$ such that $E_{i,j} \in I$.*

Proof. If there is a non-zero element $H \in \mathfrak{h}$ contained in I, then for some i we have $z_{i+1}(H) - z_i(H) \neq 0$. Since $H \in I$, the bracket $[E_{i.i+1}, H] \in I$. It is a non-zero multiple of $E_{i,i+1}$. Scaling proves that $E_{i,i+1} \in I$.

Otherwise there is an element $X \in I$ and (i,j), with $i \neq j$ such that the (i,j) entry in X is non-zero. Fix a non-zero element $H \in \mathfrak{h}$ such that $z_k(H) = z_\ell(H)$ for all $k, \ell \neq i$. Then $[H,X] \in I$ and $[H,X]$ consists of a matrix made of the sum of a multiple of the i^{th}-row of X minus the same multiple of its i^{th} column. Now let H' be a non-zero element in $\mathfrak{h}$ such that $z_k(H') = z_\ell(H')$ for all $k, \ell \neq j$. Then $[H', [H,X]] \in I$ and consists of a matrix whose (i,j) entry is $a_{i,j} \neq 0$ and whose only other possible non-zero entry is the j,i entry with coefficient $a_{j,i}$. Then twice this element plus the bracket of $H_{i,i} - H_{j,j}$ with it is contained in I and has only the (i,j) entry non-zero. □

At this point we have shown that there is (i,j) such that $E_{i,j} \in I$. Since $[E_{i,j}, E_{j,k}] = E_{i,k}$ for $k \neq i$ shows that $E_{i,k} \in I$ for every $k \neq i$. Since $[E_{i,k}, E_{\ell,i}] = -E_{\ell,k}$ for $\ell \neq k$ shows that for all $i \neq j$ the matrix $E_{i,j} \in I$. Lastly, $[E_{i,i+1}, E_{i+1,i}] = H_{i,i} - H_{i+1,i+1} \in \mathfrak{h}$. As i ranges from 1 to $n-1$ hese elements form a basis for $\mathfrak{h}$. This proves that a $\mathbb{C}$-basis for $\mathfrak{sl}(n,\mathbb{C})$ is contained in I and hence $I = \mathfrak{sl}(n,\mathbb{C})$. □

We list here some of its important properties of $\mathfrak{sl}(n,\mathbb{C})$, and their analogues for general semi-simple algebras.

(•) There is a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(n,\mathbb{C})$. Every semi-simple algebra has a Cartan subalgebra and it is unique up to the adjoint action.

($\bullet$) The action of the Cartan subalgebra decomposes $\mathfrak{sl}(n,\mathbb{C})$ into $\mathfrak{h}$ plus a collection of 1-dimension subspaces $L_{i,j}$. For a general semi-simple Lie algebra $\mathfrak{g}$, the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ decomposes it as $\mathfrak{g} = \mathfrak{h} \oplus_\alpha L_\alpha$ with the L_α being a one-dimensional joint eigenspace for $\mathfrak{h}$ with eigenvalue $\alpha\colon \mathfrak{h} \to \mathbb{C}$. The linear map α is the *root* and L_α is the *root space* associated to the root α.

($\bullet$) In the case of $\mathfrak{sl}(n,\mathbb{C})$ the subspace $\mathfrak{h}$ is the zero eigenspace for the $\mathfrak{h}$ action, and each $L_{i,j}$ corresponds to a 1-dimensional eigenspace with joint eigenvalue $\alpha_{i,j} = z_i - z_j\colon \mathfrak{h} \to \mathbb{C}$. Of course $\alpha_{j,i} = -\alpha_{i,j}$. In general the roots α are distinct and non-zero and th set of roots is invariant under multiplication by -1.

($\bullet$) For i,j,k distinct, we have $[E_{i,j}, E_{j,k}] = E_{i,k}$, $[E_{i,j}, E_{k,i}] = -E_{k,j}$, and $[E_{i,j}, E_{j,i}] \in \mathfrak{h}$. For all other pairs $[E_{i,j}, E_{r,s}] = 0$. The statement for a pairs of roots in a general semi-simple Lie algebra is that if α, α', and $\alpha+\alpha'$ are roots then $[L_\alpha, L_{\alpha'}] = L_{\alpha+\alpha'}$. Also, if $\alpha' = -\alpha$, then $[L_\alpha, L_{\alpha'}]$ is nonzero and contained in $\mathfrak{h}$. If neither of these conditions holds for the roots α and α', then $[L_\alpha, L_{\alpha']} = 0$.

($\bullet$) The bracket $[L_{i,j}, L_{j,i}] \in \mathfrak{h}$ and these elements span $\mathfrak{h}$. Also, we have$[[L_{i,j}, L_{j,i}], L_{i,j}] \neq 0$. The statement for a general semi-simple Lie algebra is $[L_\alpha, L_{-\alpha}]$ is a one-dimensional subspace of $\mathfrak{h}$ and as α varies over all roots, these subspaces generate $\mathfrak{h}$ and $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$.

($\bullet$) The 3-dimensional Lie algebra $L_{i,j} \oplus L_{j,i} \oplus [L_{i,j}, L_{j,i}]$ is the Lie algebra of $\mathfrak{sl}(2,\mathbb{C})$ and is the Lie algebra of $SL(2,\mathbb{C}) \subset SL(n,\mathbb{C})$ by the embedding induced by the inclusion of $\mathbb{C}^2 \to \mathbb{C}^n$ whose image is the sum of the i^{th} and j^{th} coordinate axes. For a general semi-simple Lie algebra, for any root α, the subspace $L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$

2.1 Split Real Form of $\mathfrak{sl}(n,\mathbb{C})$ and general Semi-Simple Lie Algebras

By a *real structure* for a complex Lie algebra $\mathfrak{g}$ we mean a real Lie algebra $\mathfrak{g}_\mathbb{R}$ togther with an isomorphism of $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} \cong \mathfrak{g}$. Another way to say this is we have a complex anti-linear involution $X \mapsto \overline{X}$ that commutes with the bracket. Then setting $\mathfrak{g}_\mathbb{R}$ equal to the fixed points of this involution produces a real Lie subalgebra of $\mathfrak{g}_\mathbb{R}$ witjh $\mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g}$.

In the standard presentation of $\mathfrak{sl}(n,\mathbb{C})$ it comes equipped with a 'natural' real structure where the anti-involution is simply the usual conjugation of complex matrices. It is clear that the brackets are real with respect to this structure (i.e., commute with conjugation). Of course the real Lie algebra is $\mathfrak{sl}(n,\mathbb{R}) \subset \mathfrak{sl}(n,\mathbb{C})$.

This real form $\mathfrak{sl}(n,\mathbb{R})$ is called the *split real form* of $\mathfrak{sl}(n,\mathbb{C})$. Real split forms are characterized by the fact that (i) the Cartan subalgebra $\mathfrak{h}$ of the complex Lie group is real (i.e., invariant under conjugation) and (ii) the eigenvalues of the conjugation action of the real subspace $\mathfrak{h}_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{C}}$ are real, i.e. the roots $\alpha\colon \mathfrak{h} \to \mathbb{C}$ are real, that is to say commute with conjugation. The real Lie group then decomposes under the adjoint action of $\mathfrak{h}_{\mathbb{R}}$ as

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus_\alpha L_{\mathbb{R},\alpha}$$

with $L_{\mathbb{R},\alpha} = L_\alpha \cap \mathfrak{g}_{\mathbb{R}}$.

Theorem 2.5. *Every complex semi-simple Lie algebra has a real split form.*

We shall not prove this result (I imagine Peter Woit will talk about it next semester). It is not deep but it does require a study of the way the roots are related to each other.

3 The Compact form of $\mathfrak{sl}(n,\mathbb{C})$ and other Semi-Simple Lie Algebras

There is another real form of $\mathfrak{sl}(n,\mathbb{C})$. Consider $SU(n) \subset SL_n(\mathbb{C})$. This is a real subgroup. Its Lie algebra $\mathfrak{su}(n)$ is $A \in \mathfrak{sl}(n,\mathbb{C})$ satisfying $\overline{A}^{tr} = -A$. This leads us to consider the anti-involution $A \mapsto -\overline{A}^{tr}$. Clearly, this is an involution and since $A \mapsto -A^{tr}$ is complex linear, this involution is complex anti-linear. Its fixed points are $\mathfrak{su}(n)$. Not only is the fixed set (the real subspace) a sub Lie algebra, but also the bracket commutes with this involution since

$$[A^{tr}, B^{tr}] = A^{tr}B^{tr} - B^{tr}A^{tr} = (BA)^{tr} - (AB)^{tr} = -[A,B]^{tr}$$

$$[-A,-B] = -(-[A,B])$$

and

$$[\overline{A},\overline{B}] = \overline{[A,B]}.$$

Together these imply

$$[-\overline{A}^{tr}, -\overline{B}^{tr}] = -\overline{[A,B]}^{tr}.$$

This real form has the property that it is the Lie algebra of a compact Lie group, namely $SU(2n)$.

Theorem 3.1. *Eery semi-simple Lie group* $\mathfrak{g}$ *has a compact real form.*

Proof. (sktech) We prove this assuming the existence of a split real form $\mathfrak{g}_{\mathbb{R}}$ for $\mathfrak{g}$. Let $\mathfrak{h}_{\mathbb{R}}$ be the real points of $\mathfrak{h}$ under the involution associated with the real split form.

There is a general theorem about the roots of a semi-simple algebra. Given a Cartan subalgebra and its roots. It is possible to divide the roots into two sets: positive roots and negative roots so that each pair $\{\alpha, -\alpha\}$ has exactly one member that is 'positive' and one that is 'negative'. Then given any set of generators $X_\alpha \in L_\alpha$ there is an automorphism of the Lie algebra that is -1 on the Cartan and for every positive root α sends X_α to $X_{-\alpha}$. (It does not necessarily send $X_{-\alpha}$ to X_α.) Choosing the X_α to be in the split real form gives a real automorphism $\varphi\colon \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$. In the case of $\mathfrak{sl}(n, \mathbb{C})$ this involution is $A \mapsto -A^{tr}$.

Claim 3.2. φ *is an involution.*

Proof. Consider a $X_\alpha \in L_\alpha$ and let $Y_\alpha \in L_{-\alpha} = \varphi(X_\alpha)$. Then set $H_\alpha = [X_\alpha, Y_\alpha] = H \in \mathfrak{h}$. Since φ is an automorphism, we have

$$[\varphi(X_\alpha), \varphi(Y_\alpha)] = \varphi(H_\alpha) = -H_\alpha.$$

We conclude that $[\varphi(Y_\alpha), Y_\alpha] = H_\alpha =]X_\alpha, Y_\alpha]$. Since $\varphi(Y_\alpha) \in L_\alpha$, it follows that $\varphi(Y_\alpha) = X_\alpha$. This proves that on the root spaces $\varphi^2 = \mathrm{Id}$. Since $\varphi|_{\mathfrak{h}} = -1$, its square is also the identity on $\mathfrak{h}$. □

Now consider the composition $\varphi \circ \sigma = \sigma \circ \varphi$ where σ is anti-linear involution determined by the split real form. This is an anti-linear involution. Let $\mathfrak{g}_c$ be its fixed subspace. It is another real from for $\mathfrak{g}$. It has the property that its intersection with $\mathfrak{h}$ is $i\mathfrak{h}_{\mathbb{R}}$.

Now let us introduce the *quadratic Casimir* operator. This is a bilinear form on $\mathfrak{g}$ defined by

$$B(X, Y) = \mathrm{Trace}(ad(X) \circ ad(Y)).$$

Clearly this is a symmetric complex bilinear form on $\mathfrak{g}$. If $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$, then the restriction of B to $\mathfrak{g}_0$ is a real symmetric bilinear form.

Claim 3.3. *If* $\mathfrak{g}_c$ *is a compact real form for* $\mathfrak{g}$ *then the restriction to* $\mathfrak{g}_c$ *of* B *is negative definite.*

Proof. Let $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$. Since $\mathfrak{h}_\mathbb{R}$ is the split real form $\alpha\colon \mathfrak{h}_\mathbb{R} \to \mathbb{R}$ for each root. From the fact that $\mathfrak{h}_c = i\mathfrak{h}_\mathbb{R}$ we see that $\alpha\colon \mathfrak{h}_c \to i\mathbb{R}$.

$$\text{Trace}(ad(h) \circ ad(h)) = \sum_\alpha \alpha(h)^2.$$

Fix $h \in \mathfrak{h}_c$. Then

$$B(h,h) = \sum_\alpha \alpha(h)^2 \leq 0.$$

To see that $B(h,h) < 0$ we need only show that there is a root α with $\alpha(h) \neq 0$. But if $\alpha(h) = 0$ for all roots α, then $[h, L_\alpha] = 0$ for all root spaces L_α. Of course $[h, \mathfrak{h}] = 0$. Together these prove that h is in the center of $\mathfrak{g}$. But the center is an ideal of $\mathfrak{g}$ and hence must be zero, showing that $h = 0$. This proves that the $Tr(h \circ h) < 0$ for all $h \in \mathfrak{h}_0$.

Since any root vector $X_\alpha \in L_\alpha$ maps the eigenspace with eigenvalue a to one with eigenvalue $a + \alpha \neq a$, it follows that $\text{Trace}(X_\alpha \circ X_\beta) = 0$ unless $\beta = -\alpha$. Thus, under B the two-dimensional spaces $V_\alpha = L_\alpha \oplus L_{-\alpha}$ are pairwise orthogonal and orthogonal to $\mathfrak{h}_\mathbb{R}$. Fix $X_\alpha \in L_\alpha \cap \mathfrak{g}_\mathbb{R}$ and let $Y_\alpha = \varphi(X_\alpha) \in L_{-\alpha} \cap \mathfrak{g}_\mathbb{R}$. Then the basis for $(L_\alpha \oplus L_{-\alpha}) \cap \mathfrak{g}_c$ is $\{X_\alpha - Y_\alpha, iX_\alpha + iY_\alpha\}$. We hav just seen that these two dimensional spaces are mutually orthogonal under B and also each is orthogonal to $i\mathfrak{h}_\mathbb{R}$.

Since B is negative definite on $i\mathfrak{h}_\mathbb{R}$ we need only see that B is negative definite on each of these two dimensional spaces. We compute

$$\begin{aligned}\text{Trace}((a(ad(X_\alpha) - ad(Y_\alpha)) + b(ad(iX_\alpha) + ad(iY_\alpha)))^2) = \\ \text{Trace}(((a+b)ad(X_\alpha) + (-a+ib)ad(Y_\alpha))^2).\end{aligned}$$

Since $\text{Trace}(X_\alpha^2) = \text{Trace}(Y_\alpha^2) = 0$, the above trace is

$$(a+ib)(-a+ib)(\text{Trace}(ad(X_\alpha) \circ ad(Y_\alpha) + ad(Y_\alpha)_\circ ad(X_\alpha)) = -2(a^2+b^2)\text{Trace}(ad(X_\alpha) \circ ad(Y_\alpha)).$$

The computation of all finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ representations shows that on any irreducible representation V we have $\text{Trace}_V(X_\alpha \circ Y_\alpha) \geq 0$ and the trace is 0 only for the trivial representation. The adjoint representation decomposes under the $\mathfrak{sl}(2,\mathbb{C})$ generated by X_α and Y_α as a sum of irreducible representations, one of which is the two-dimensional representation. It follows that $\text{Trace}(ad(X_\alpha) \circ Y_\alpha) > 0$.

This completes the proof that B is negative definite on the two dimensional subspace associated with each pair $\pm\alpha$ of roots, and hence is negative definite on $\mathfrak{g}_c$. □

Next, we need to show that the adjoint action of $\mathfrak{g}_c$ preserves B:

Claim 3.4.

$$B([X,Y],Z) + B(Y,[X,Z]) = 0.$$

Proof. We compute:

$$\mathrm{Trace}(ad([X,Y] \circ ad(Z)) = \mathrm{Trace}(ad(X)ad(Y)ad(Z) - ad(Y)ad(X)ad(Z))$$

$$\mathrm{Trace}(adY \circ ad([X,Z])) = \mathrm{Trace}(ad(Y)ad(X)ad(Z) - ad(X)ad(Z)ad(Y)).$$

The sum of these two terms is

$$\mathrm{Trace}(ad(X)ad(Y)ad(Z)) - \mathrm{Trace}(ad(X)ad(Z)ad(Y)),$$

which vanishes since $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$. □

Let G_c be the adjoint form of $\mathfrak{g}_c$; that is to say $G_c \subset GL(\mathfrak{g}_c)$ is a subgroup $Auto(\mathfrak{g}_c)$ with Lie algebra the image under the adjoint representation of $\mathfrak{g}_c \to \mathfrak{g}l(\mathfrak{g}_c)$. This implies that G_c is a subgroup of the orthogonal group of B, which is a compact group. On the other hand G_c is the real form of the complex group $ad(G)$. It follows from the semi-simplicity of $\mathfrak{g}$ that G is the component of the identity in $Aut(\mathfrak{g})$. [We shall not prove this, it again follows from a study of the structure of the roots of $\mathfrak{g}$.] Thus, that the adjoint representation sends $\mathfrak{g}$ isomorphically to $\mathrm{End}(\mathfrak{g})$ preserving quadratic form B. Hence, the image under the adjoint map of the real form, $\mathfrak{g}_c$, is the Lie subalgebra of $\mathrm{End}(\mathfrak{g}_c)$ preserving the restriction of B to $\mathfrak{g}_c$. Thus, G_c is the component of the identity of the group of B-orthogonal transformations of $\mathfrak{g}_c$. Since $B|_{\mathfrak{g}_c}$ is negative definite, this group is compact. □

4 Complete reducibility of representations of $\mathfrak{s}l(n,\mathbb{C})$ and more general Semi-Simple Lie Algebras

Theorem 4.1. *Any finite dimensional complex representation of* $\mathfrak{sl}(n,\mathbb{C})$ *is completely reducible,*

Proof. Let V be a (complex) representation of $\mathfrak{s}l(n,\mathbb{C})$ and suppose that $W \subset V$ is an $\mathfrak{s}l(n,\mathbb{C})$-invariant subspace. Restricting to the compact real form gives us a representation of $\mathfrak{s}u(n)$ and by the exponential mapping a complex linear representation $SU(n) \times V \to V$. Of course, $SU(n)$ is compact and W is an $SU(n)$-invariant subspace. Using a Haar measure and integrating we introduce an $SU(n)$-invariant Hermitian inner product on V. The orthogonal complement $W^\perp$ is invariant under $SU(n)$ and hence under

$\mathfrak{su}(n)$ which acts on it by complex linear endomorphisms. Since $\mathfrak{su}(n)\otimes_{\mathbb{R}}\mathbb{C} = \mathfrak{sl}(n,\mathbb{C})$, it follows that $W^{\perp}$ is also invariant under $\mathfrak{sl}(n,\mathbb{C})$.

Now the usual induction argument shows complete reducibility of the $\mathfrak{sl}(n,\mathbb{C})$. □

Remark 4.2. Given the claim that all semi-simple complex Lie algebras have compact real forms complete reducibility holds for representations of these Lie algebras as well.

In fact, a wider class of complex Lie algebras, called *reductive* Lie algebras have a compact form, and hence any finite dimensional representation of any reductive group that induces a representation of the compact Lie group is completely reducible. The simplest reductive group is $\mathbb{C}^*$. Its compact form is S^1 and its split real form is $\mathbb{R}^*$. Using the compact real form and arguing as above gives another proof that any finite dimensional representation of $\mathbb{C}^*$ is completely reducible.

5 From Compact Groups to Semi-Simple Lie algebras

What is often presented in mathematical contexts is a compact Lie group, e.g. the Orthogonal group. From this group we can directly produce a reductive complex Lie algebra and then a sermi-simple Lie algebra.

Definition 5.1. Let $\mathfrak{g}$ be a Lie algebra. Its *adjoint form*, $ad(\mathfrak{g})$, is the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ that is the image of the adjoint map $ad\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The *Adjoint form* of $\mathfrak{g}$ is the quotient of the simply connected Lie group G with Lie algebra $\mathfrak{g}$ by its center. The Lie algebra of the Adjoint form of $\mathfrak{g}$ is the adjoint form of $\mathfrak{g}$.

Theorem 5.2. *Let $\mathfrak{g}$ be a Lie algebra. Suppose the simply connected Lie group G with Lie algebra $\mathfrak{g}$ is compact and has finite center. Then:*

- *The adjoint action of $\mathfrak{g}_{\mathbb{C}}$ is injective and completely reducible meaning that there is a decomposition $\mathfrak{g}_{\mathbb{C}} = \oplus_{i\in I} V_i$ where the V_i are irreducible representations of $\mathfrak{g}_{\mathbb{C}}$ under the adjoint action.*
- *The V_i are ideals in the Lie algebra $\mathfrak{g}_{\mathbb{C}}$.*
- *For $i \neq j$, the subalgebras V_i and V_j commute with each other in the sense that $[V_i, V_j] = 0$.*

- *Each V_i is a simple Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ is a semi-simple algebra.*
- *The Lie algebra of $\mathfrak{g}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$.*

Proof.

Claim 5.3. *The center of $\mathfrak{g}$ is trivial.*

Proof. Since the center of G is discrete, the Lie algebra of G and of its adjoint form are the same. That is to say the adjoint for $\mathfrak{g}$ is $\mathfrak{g}$ itself, which means that the center of $\mathfrak{g}$ is trivial. □

From this we see that the center of $\mathfrak{g}_{\mathbb{C}}$ is trivial, and hence that the adjiont representation of $\mathfrak{g}_{\mathbb{C}}$ is injective. That is to say, $\mathfrak{g}_{\mathbb{C}}$ is its own adjoint form. A real form of $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{g}$ and a real form of the adjoint representation $\mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and the real form of the representation exponentiates to give a linear representation of the adjoint form of G. Since this group is compact, the standard argument shows that the adjoint representation of $\mathfrak{g}_{\mathbb{C}}$ on itself is completely decomposable. That is to say $\mathfrak{g}_{\mathbb{C}} \cong \oplus_i V_i$ where the V_i are irreducible sub-representations of $\mathfrak{g}_{\mathbb{C}}$ acting by the adjoint representation..

Claim 5.4. *The V_i are ideals in $\mathfrak{g}_{\mathbb{C}}$ and commute with each other.*

Proof. The fact that V_i is a submodule for the adjoint representation means for any $X \in \mathfrak{g}_{\mathbb{C}}$ and any $v \in V_i$ we have $[X, v] \in V_i$. That is the statement that V_i is an ideal. Now suppose that $v_i \in V_i$ and $v_j \in V_j$ for $i \neq j$. Then $[v_i, v_j] \in V_j$ and $[v_i, v_j] = -[v_j, v_i] \in V_i$. This proves that $[v_i, v_j] = 0$ and proves the various V_i are commuting ideals. In particular the V_i are sub-algebas of $\mathfrak{g}_{\mathbb{C}}$. □

Next suppose that $J_i \subset V_i$ is an ideal for the adjoint action of V_i on itself. Since $[V_i, V_j] = 0$ for $i \neq j$ and J is an ideal for V_i acting on itself, J is an ideal for the action of $\mathfrak{g}_{\mathbb{C}}$. Since V_i is irreducible as a $\mathfrak{g}_{\mathbb{C}}$-module this implies that either $J = 0$ or $J = V_i$. This shows that the Lie algebra V_i has no non-trivial ideals, and hence provided that $\dim(V_i) > 1$, V_i is a simple algebra.

It remains only to show that none of the V_i are one-dimensional. If $\dim(V_i) = 1$, then $[V_i, V_i] = 0$, since we have already seen that $[V_j, V_i] = 0$ for all $j \neq i$, it follows that $[V_i, \mathfrak{g}_{\mathbb{C}}] = 0$, which means that V_i is in the center of $\mathfrak{g}_{\mathbb{C}}$. But we have already shown that the center of $\mathfrak{g}_{\mathbb{C}}$ is trivial. This proves that $\mathfrak{g}_C$ is semi-simple. □

Corollary 5.5. *Let* $\mathfrak{g}$ *be a Lie algebra and suppose that the Adjoint form of this Lie algebra is a compact group with finite fundamental group, e. g.,* $\mathfrak{s}o(n)$ *or* $\mathfrak{s}u(n)$*. Then* $\mathfrak{g}_{\mathbb{C}}$ *is semi-simple.*

It turns out that there are four infinite series of simple Lie algebras and 5 exceptional Lie simple Lie algebras. The simply series are:

- $\mathfrak{s}l(n, \mathbb{C})$ with compact form $SU(n)$
- $\mathfrak{s}o(2n)_{\mathbb{C}}$ with compact form $SO(2n)$
- $\mathfrak{s}o(2n+1)_{\mathbb{C}}$ with compact form $SO(2n+1)$.
- The complex symplectic Lie algebras $\mathfrak{s}p(2n)_{\mathbb{C}}$ with compact form the intersection of $SP(2n)_{\mathbb{C}} \cap SU(2n)$.

For these four series we immediately have the complex group is semi-simple and any representation is completely reducible.