Lie Groups: Fall, 2022
Syllabus

August 23, 2022

Lecture I. The Basics

(i) Definition of Real and Complex Lie groups, of linear algebraic groups over a field.

(ii) Examples of Lie Groups: Permutation groups, groups of symmetries of regular polygon, of regular polyhedra, of the spheres, of quadratic forms on $\mathbb{R}^n$, of the upper half-plane, of hyperbolic space of any dimension,

(iii) Definition of a Lie algebra over a field.

(iv) Examples of Lie algebras – $GL(n),\ SL(n),\ O(n)$

Lecture II. Real and Complex Lie Groups and Lie Algebras and the Adjoint Actions

(i) Real Lie Groups and sub Groups and one-one immersed 'subgroups'

(ii) Quotients manifolds of subgroups; Quotient Lie group of normal Lie subgroup

(iii) $O(n)$ and $SO(n)$

(iv) one-one maps of Lie groups whose images are not subgroups

(v) Definition of Actions and linear representations

(vi) The Adjoint action; computation for $GL(n,\mathbb{R})$ and its subgroups

(vii) Left-invariant ector fields and the Lie algebra of a general group

(viii) Funcrtorality of the Lie algebra of a Lie group.

Lecture III. Relations of Lie Groups and Lie Algebras

(i) Universal Enveloping Algebra of a Lie Algebra; Poincaré-Birkhoff-Witt Theorem

(ii) The Exponential Mapping

(iii) Definition of Local Lie Groups; Lie algebra of a Local Lie Group; local Lie subgrops of a Lie group

(iv) Extension of local Lie subgroups of a Lie group to a Lie group

(vi) The germ of a local Lie subgroup determined by a Lie Algebra.
(a) The Baker-Hausdorff-Campbell Formula
(b) Convergence for finite dimensional real Lie algebras
(c) Main Existence and Uniqueness theorem
(vii) The simply connected Lie group of a finite dimensional Lie Algebra
(a) Ado's Theorem on finite dimensional faithful representations
(b) All Lie groups with a given Lie algebra are quotients of the simply connected Lie group by a discrete subgroup of its center.

Lecture IV. Clifford Algebras and Spin Grops
(i) Clifford Algebra of a Non-degenerate Real Quadratic form
(ii) Basic relationships between various Clifford Agebras
(iii) Low dimensional examples
(iv) Definitions of $Pin(V,Q)$ and $Spin(V,Q)$
(v) Double covering $Spin(V,Q) \to SO(V,Q)$

Lecture VI. Finite dimensional Representations of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{o}(3)$
(i) Presentation of the Lie Algebra $\mathfrak{sl}(2,\mathbb{C}$.
(ii) Classification of its finite dimensional complex representations
(iii) Isomorphism $\mathfrak{o}(3) \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathfrak{sl}(2,\mathbb{C})$
(iv) Classification of finite dimensional real representations of $\mathfrak{o}(3)$.

Lecture V. The Heisenberg Group and its Representations

Lecture VII. Compact Lie Groups
(i) Haar measure and integration over compact groups
(ii) G-invariant Riemannian metrics,
(iii) Compete reducible of finite dimensional representations

Lecture VIII. Unitary Representations of compact Lie Groups
(i) Peter-Weyl Theorem on complete reducibility
(ii) Fourier series
(iii) spherical harmonics