Prerequisites for Lie Groups: Fall, 2022

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1 Basics of Group Theory

The following definitions and exercises should be well-known to you. If not, work through them, consulting an elementary text on abstract group theory as needed.

Definition 1.1. A *group* is a set G with a *multiplication* $m: G \times G \to G$, written $m(g_1, g_2) = g_1 g_2$, that:

1. is associative $((g_1 g_2) g_3) = (g_1 (g_2 g_3))$, for all $g_1, g_2, g_3 \in G$,
2. has a unit $e \in G$ such that $eg = ge = g$ for all $g \in G$, and
3. has inverses, for every $g \in G$, there is $g^{-1} \in G$ with $gg^{-1} = g^{-1}g = e$.

A *subgroup* of a group G is a subset of the underlying set of G that contains the identity element of G and is closed under multiplication and taking inverses. If $H \subset G$ is a subgroup then we define the set or left cosets of H to be the set of equivalence classes of elements of G where $g \cong g'$ if there is $h \in H$ with $gh = g'$. Similarly, define the right cosets of H.

Given two groups G and H there is a product group $G \times H$. Its underlying set is the set-theoretic product of the underlying sets of G and H. The multiplication is given by $(g_1, h_1)(g_2, h_2) = (g_1 g_1, h_1 h_2)$.

Exercise 1. Show that the identity element is unique, and show that for each $g \in G$ its inverse is unique.
Exercise 2. Show that this multiplication defines a group structure. Identity the identity element and the inverse of (g, h).
3. Extend this to an arbitrary product of any set of groups.
4. Let S be a set. Show the set of bijective functions forms a group under composition: namely $g_1 g_2$ is the composition of first applying the bijection g_2 and then applying the bijection g_1. This group is denoted $\mathrm{Aut}(S)$. Show

that if S is a set of n elements then its group of bijections $\mathrm{Aut}(S)$ is a finite group of order n.

Definition 1.2. If G and H are groups a *homomorphhism* $\psi\colon G \to H$ is a set function from the underlying set of G to that of H that preseves the multiplication and identities. Show that the kernel of a homorphism (the elements sent to the idenity) is a subgroup of G.

An action of a group on a set S is a homomorphism from $G \to \mathrm{Aut}(S)$. That is to say there is a function $G \times S \to S$ written $(g, s) \mapsto gs$ such that $g_1(g_2 s) = (g_1 g_2)s$ and $es = s$ for all $s \in S$.

Let G be a group. An automorphism of G is an automorphism of the underlying set that is a homomorphism of the group to itself.

Exercise 5. Show that if $\psi\colon G \to G$ is an automorphism, then the inverse function $\psi^{-1}\colon G \to G$ is also an automorphism of G. Conclude that the set of automorphisms of G forms a group under composition. This group is denoted $\mathrm{Aut}(G)$.

Exercise 6. Show that there is a natural homomorphism $G \to \mathrm{Aut}(G)$ given by sending $g \in G$ to the automorphism that sends $g' \in G$ to $gg'g^{-1}$. Show that the kernel of this action is the *center* of G, i.e., the element $z \in G$, that commute with eery element of G, in the sense that $zg = gz$ for all $g \in G$.

A subgroup $H \subset G$ is *normal* if the conjugation action of G on itself stabilizes H, i.e., maps H to itself. Show that the center of G is a normal subgroup.

Exercise 7. Show that the kernel of a homomorphism $\psi\colon G \to K$ is a normal subgroup. Conversely, show that if $H \subset G$ is a normal subgroup then there is an induced multiplication on the set of left cosets G/H producing a group structure on G/H with the property that the natural quotient map $G \to G/H$ is a surjective homomorphism with kernel H.

A group is *abelian* if $g_1 g_2 = g_2 g_1$ for all elements $g_1, g_2 \in G$.

Exercise. 8 Show that every subgroup of an abelian group is normal.

A group N is *nilpotent* if there is a series of normal subgroups of N

$$\{e\} \subset N_k \subset N_{k-1} \subset \cdots \subset N_1 = N$$

with the property that for each j, N_j/N_{j+1} is contained in the center of N/N_{j+1}.

Example. Strictly upper triangular integral $n \times n$ matrices form a nilpoint group under matrix multiplication.

2 Basics of Algebraic Geometry

Fix a field K.. Then $K[x_1, \ldots, x_n]$ of polynomial functions in n-variables is the ring of *algebraic functions* on K^n. A *subvariety* of K^n (also called an *affine algebraic variety*) is the locus of vanishing of a given collection of polynomials. It is an easy exercise to show that any subvariety of K^n is the locus of vanishing of a finite set of polynomials. Hence, the sub-varieties are the closed subsets of a topology on K^n, called the *Zariski topology*. If V is a sub-variety of K^n and $I(V) \subset K[x_1, \ldots, x_n]$ is the ideal of functions vanishing identically on V, then we define the *ring of algebraic functions* on V to be $K[x_1, \ldots, x_n]/I(V)$. Thus, the algebraic functions on V are exactly the restrictions of algebraic functions on all of K^n to V. The inclusion of an affine sub-variety into K^n is then an algebraic mapping (i.e., one that pulls algebraic functions on K^n bak to algebraic functions on V).

For any non-zero function $f \in K[x_1, \ldots, x_n]$, let D_f be the *divisor defined by* f, meaning the locus where f vanishes. Without loss of generality it suffices to assume that f is not a power greater than 1 of another polynomial. Then the complement $K^n \setminus D_f$ is itself an affine variety under the embedding $U \subset K^n \times K$ where the map on the first factor is the inclusion and the map on the second factor is $1/f$. Under this embedding the ring of polynomial functions on U is the ring generated by the restrictions of the x_i and f^{-1} to U.

It is not true that every Zariski open subset of K^n is an affine variety. For example $K^2 \setminus \{(0,0)\}$ is not an affine algebraic variety. The point is that any mapping $K^2 \setminus \{(0,0)\} \to K$ that is algebraic is the restriction of a polynomial of two variables, so any algebraic map $K^2 \setminus \{(0,0)\} \to K^n$ extends over K^2. [To make this argument rigorous, one must introduce the notion of more general varieties as ringed spaces.]

3 Basics of Differential Topology

Two central theorems from Calculus are:

Theorem 3.1. *(Inverse Function Theorem) Let $U \subset \mathbb{R}^n$ be an open subset and $\Phi\colon U \to \mathbb{R}^n$ a smooth function. If for some $x \in U$ we have $D\Phi_x\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then there is a neighborhood $V \subset U$ of x so that $\Phi|_V\colon V \to \mathbb{R}^n$ is a diffeomorphism of V onto an open subset of $\mathbb{R}^n$.*

A direct application of the Inverse Function Theorem is the following:

Theorem 3.2. *(The Implicit Function Theorem) Let $U \subset \mathbb{R}^n$ be an open subset and $M \subset U$ a closed subset of U given by the vanishing of a smooth funcion $\Phi\colon U \to \mathbb{R}^k$. If at a point $x \in M$, the differential $D\Phi_x\colon \mathbb{R}^n \to \mathbb{R}^k$ is surjective, then after re-numbering the coordinates of $\mathbb{R}^n$, we have $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$ with $x = (x_1, x_2)$, a neighborhood V_1 of x_1 in $\mathbb{R}^{n-k}$, a neighborhood V_2 of $x_2 \in \mathbb{R}^k$, and a smooth function $\psi\colon V_1 \to V_2$ so that $M \cap (V_1 \times V_2)$ is the graph of ψ. In particular, projection of $M \to \mathbb{R}^{n-k}$ is a local diffeomorphism near x.*

Now suppose that $U \subset \mathbb{R}^n$ is an open set and $\Phi\colon U \to \mathbb{R}^k$ is a smooth function whose differential is of rank k at every $x \in M$. Then the inverses of the local projections from the implicit function theorem given smooth maps from open subsets of $\mathbb{R}^{n-k}$ to M, homeomorphisms whose images cover M and with the property that on the overlap of two of these maps the composition of one followed by the inverse of the other is a diffeomorphism between open subsets of $\mathbb{R}^{n-k}$.

This leads us to the abstract definition of a smooth manifold of dimension r. It has a covering by open subsets each equipped with a homeomorphism to an open subset of $\mathbb{R}^r$ so that on the overlaps the composition of the inverse of one followed by the other is a diffeomorphism between open subsets of $\mathbb{R}^r$. Two such coordinate atlases define the same smooth structure on the space if the overlaps between the two sets of charts satisfy the same condition (i.e., if the union of the two sets of smooth charts is another set of smooth charts.

Smooth manifolds are the objects of a category, the *smooth category.* The morphisms are smooth maps, i.e., maps which with respect to the local smooth coordinate systems on domain and range are smooth (i.e., C^∞) in the usual sense.

This category has finite products and sums (disjoint unions)

Associated to any smooth manifold is its tangent bundle. The fiber over $x \in M$ is the tangent space to M at x. This can be thought of as the dual space to the quotient of the ideal functions on M vanishing at x modulo the square of this ideal. It can also be thought of equivalence classes of curves passing through x at parameter value 0, where two such curves are equivalent if they agree to first order at x. Given such a curve $\gamma(t)$ and a function φ vanishing at x the pairing is

$$\langle \gamma, \varphi \rangle = \frac{d\varphi \circ \gamma(t)}{dt}|_{t=0}.$$

Using local coordinates $(x^1, \ldots, x^n)$ a basis for the tangent space at x is $\{(\partial/\partial x^i)|_x\}$. Thus, using local coordinate systems we give a product structure to the union over a coordinate patch of the tangent spaces at the various points The product structure identifies the standard basis of $\mathbb{R}^n$ with $\{(\partial/\partial x^i)|_x\}$. Of course, if we change the local coordinates this product structure changes but it changes by a smooth vary automorphism, i.e., a smooth map from the overlap of the two coordinate systems to $GL(n, \mathbb{R})$.

Thus, these local trivializations fit together to define the structure of a smooth vector bundle over the manifold, called *its tangent bundle*, and denoted TM. Its fiber at $x \in M$ is T_xM.. A smooth map $f\colon M \to N$ lifts to a map of vector bundles $Df\colon TM \to TN$.