Lie Groups: Fall, 2022
Lecture VIC: Killing Form and Semi-simpliciity

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1 The Main Result

Here is the main result we shall establish.

Theorem 1.1. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. The following are equivalent.*

- *The radical of $\mathfrak{g}$ is trivial.*
- *The Killing form $B\colon \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$ given by $B(X,Y) = Tr(ad(X)\circ ad(Y))$ is non-degenerate.*
- *$\mathfrak{g}$ is semi-simple.*

Corollary 1.2. *If $\mathfrak{g}$ is a finite dimensional, complex Lie algebra and $\mathfrak{r}$ is it sradical, then $\mathfrak{g}/\mathfrak{r}$ is semi-simple*

Proof. (that the theorem implies the corollary) If L is a finite dimensional Lie algebra with a solvable ideal I with the property that L/I is a solvable Lie algebra then L is solvable. From this it follows that the radical of $\mathfrak{g}/\mathfrak{r}$ is trivial. According to the theorem this means $\mathfrak{g}/\mathfrak{r}$ is semi-simple. □

Since a simple algebra has no non-trivial ideals and a semi-simple algebra is a commuting direct sum of simple algebras, the only ideals of a semi-simple algebra are themselves semi-simple algebras. If $\mathfrak{g}$ has a non-trivial, solvable ideal, then either that ideal is abelian or its commutator sub algebra is a non-zero nilpotent ideal. In that case the center of that nilpotent ideal is a non-zero commutative ideal. Thus, if the radical of $\mathfrak{g}$ is non-zero, then $\mathfrak{g}$ has a non-zero commutative ideal. Since a non-zero commutative algebra is a direct sum of one-dimensional algebras it is not semi-simple. This shows that the third item of Theorem 1.1 implies the first.

According to Corollary 4.3 of Lecture VIB, the null space of the Killing form is a solvable ideal of $\mathfrak{g}$. Thus, the first item of the theorem implies the second.

To complete the proof of the theorem itt remains to show that if the Killing form is non-degenerate then $\mathfrak{g}$ is semi simple. That is the content of the next two sections.

2 the Casimir Operator

In this subsection we study a Lie algebra $\mathfrak{g}$ and a finite dimensional $\mathfrak{g}$-module V with the property that $B_V(X,Y) = Tr_V(XY)$ is a non-degenerate pairing. Fundamental to this study is the notion of a Casimir operator.

2.1 Casimir operators C_V

Definition 2.1. Let $\mathfrak{g} \subset \mathfrak{g}l(V)$ be a Lie algebra. We define a right action of $\mathfrak{g}$ on the linear dual $\mathfrak{g}^*$.

$$\mathfrak{g}^* \otimes \mathfrak{g} \to \mathfrak{g}^*$$

by $(\varphi \otimes X)(Y) = \varphi(ad(X)Y)$. Direct computation shows that this is a right action.

The non-degenerate pairing B_V induces an isomorphism $\Psi\colon \mathfrak{g} \to \mathfrak{g}^*$ sending X to $B_V(X, \cdot)$.

Claim 2.2. *For $X, Y \in \mathfrak{g}$ we have $\Psi(X) \cdot Y = \Psi \cdot (ad(X)(Y))$.*

Proof. $\Psi(X) \cdot Y$ is the homomorphiusm $B_V(X, [Y, \cdot])\colon \mathfrak{g} \to \mathbb{C}$. On the other hand, $\Psi \cdot (ad(X)(Y))$ is the homomoprhism $B_V([X,Y], \cdot)$. As we have seen before, these homomorphisms are equal. □

Now we view B_V as an element in $\mathfrak{g}^* \otimes \mathfrak{g}^*$. The equation $B_V([X,Y],Z) + B_V(Y,[X,Z]) = 0$ is the statement that under the right action of $\mathfrak{g}$ on $\mathfrak{g}^* \otimes \mathfrak{g}^*$ given by $(f \otimes g) \cdot X = (f \cdot X) \otimes g + f \otimes (g \cdot X)$, the element B_V is invariant under the $\mathfrak{g}$ action. We obtain an element $(\Psi \otimes \Psi)^{-1}(B_V) \in \mathfrak{g} \otimes \mathfrak{g}$ that is invariant under the usual action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$; namely the action $X \cdot (Y \otimes Z) = [X,Y] \otimes Z + Y \otimes [X,Z]$.

Next, let us write an expression for $(\Psi \otimes \Psi)^{-1}(B_V)$. Fix. basis $\{X_i\}_i$ for $\mathfrak{g}$ and let $\{Y_i\}_i$ be the basis dual to the first one under B_V; that is to say $B_V(X_i, Y_J) = \delta(i,j)$. Then we have the algebraically dual bases $\{X_i^*\}_i$ and

$\{Y_i\}^*$ of $\mathfrak{g}^*$. Clearly, $\Psi(X_i) = Y_i^*$ and $\Psi(Y_i) = X_i^*$, and $B_V = \sum_i X_i^* \otimes Y_i^*$. Consequently,

$$(\Psi \otimes \Psi)^{-1}(B_V) = \sum_i X_i \otimes Y_i.$$

We can also view C_V as an element in the universal enveloping algebra $U(\mathfrak{g})$; namely $C_V = \sum_i X_i Y_i$. This element is invariant under the natural adjoint action of the Lie $\mathfrak{g}$ on $U(\mathfrak{g})$, given by $X \cdot a = Xa - aX$ for any $a \in U(\mathfrak{g})$ and any $X \in \mathfrak{g}$

Corollary 2.3. *Let $\mathfrak{g} \subset \mathfrak{g}l(V)$ be a subalgebra with the property that $B_V \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is non-degenerate. Then $C_V \colon V \to V$ is a homomorphism of $\mathfrak{g}$-modules and its trace is* $\dim(\mathfrak{g})$.

Proof. Since C_V is invariant under the action of $\mathfrak{g}$, for all $X \in \mathfrak{g}$ we have $XC_V = C_V X$. That is to say $X(C_V(v)) = C_V(Xv)$, or equivalently that $C_V \colon V \to V$ is a morphism of $\mathfrak{g}$-modules. Since $B_V(X_i, Y_i) = 1$, it follows that for each i, $Tr_V(X_i Y_i) = 1$, and hence $Tr_V(\sum_i X_i Y_i) = \dim(\mathfrak{g})$. □

3 Complete reducibility of $\mathfrak{g}$-modules V with B_V non-degenerate

Theorem 3.1. *Let $\mathfrak{g} \subset \mathfrak{g}l(V)$ be a sub Lie algebra. Suppose that B_V is non degenerate. Then V is completely reducible as a $\mathfrak{g}$-module.*

Proof. It suffices to prove that every $\mathfrak{g}$-submodule of V has a complementary $\mathfrak{g}$-submodule. If $\mathfrak{g} \subset \mathfrak{g}l(V)$ is the trivial algebra, then this is clear. We assume from now on that $\mathfrak{g} \subset \mathfrak{g}l(V)$ is a non-zero.

Case 1. The $\mathfrak{g}$-submodule is of codimension 1. We argue by induction on the dimension of V. When V has dimension 1 the result is trivial. Suppose that we know the result for all $\mathfrak{g}$-modules V' of dimension less than V and let $W \subset V$ be a $\mathfrak{g}$-submodule of codimension 1. Suppose that W is not simple. That is to say, W has a $\mathfrak{g}$-submodule W' different from W and $\{0\}$. Then $W/W' \subset V/W'$ is of codimension 1 and since $W' \neq \{0\}$, the dimension of V/W" is less than that of V. By induction, there is a complementary $\mathfrak{g}$-submodule $L \subset V/W'$ to W/W'. The pre-image, $\widetilde{L}$ of L in V is a $\mathfrak{g}$-submodule that contains W' as a codimension 1 submodule. Furthermore, since $W' \neq W$, the dimension of $\widetilde{L}$ is less than that of V. Hence, by induction there is a complementary $\mathfrak{g}$-module $L' \subset \widetilde{L}$ to W'. Clearly, L' is a $\mathfrak{g}$-submodule of V complementary to W.

It remains to consider the case when $W \subset V$ is a simple $\mathfrak{g}$-module. The Casimir map $C_V\colon V \to V$ is a $\mathfrak{g}$-module map and it maps W to W and is trivial on the one-dimensional quotient V/W. Since W is a simple $\mathfrak{g}$-module, $(C_V)|_W$ is multiplication by a constant. Thus, $Tr(C_V) = Tr((C_V)|_W)$. Since the dimension of $\mathfrak{g} \subset \mathfrak{g}l(V)$ is positive, it follows from Corollary 2.3 that $Tr((C_V)|_W)$ is positive and hence $C_V|_W$ is multiplication by a positive constant and hence an isomorphism. Since C_V induces the zero map from $V/W \to V/W$, it follows that $\mathrm{Ker}(C_V)$ is a complementary linear subspace to W. Since C_V is a $\mathfrak{g}$-module map its kernel is a $\mathfrak{g}$-module. This produces the complementary $\mathfrak{g}$-module to W. This completes the proof when for submodules of codimension 1.

Case 2. The $\mathfrak{g}$-submodule is of codimension > 1. Now we consider a general submodule $W \subset V$ of codimension k. If W is not simple as a $\mathfrak{g}$-module, then dividing pout by a proper submodule and arguing by induction on the dimension of V we see that, similarly to the argument in the codiemsnion 1 case, W has a complementary $\mathfrak{g}$-module.

It remains to consider the case when W is simple. Then $\mathrm{Hom}_{\mathfrak{g}}(W,W)$ is one dimensional consisting only of scalar multiplication. For $\mathfrak{g}$-modules A and B, we define a $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbb{C}}(A,B)$ by

$$X \cdot f(a) = X(f(a)) - f(Xa).$$

Then the restriction map

$$\rho\colon \mathrm{Hom}_{\mathbb{C}}(V,W) \to \mathrm{Hom}_{\mathbb{C}}(W,W)$$

is a surjective $\mathfrak{g}$-module map. In $\mathrm{Hom}_{\mathbb{C}}(W,W)$ there is the one-dimensional $\mathfrak{g}$-submodule $\mathrm{Hom}_{\mathfrak{g}}(W,W)$. The pre image $U = \rho^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W,W))$ is a $\mathfrak{g}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(V,W)$ mapping onto the trivial 1-dimensional module $\mathrm{Hom}_{\mathfrak{g}}(W,W)$. The kernel of ρ is a $\mathfrak{g}$-submaodule of U of codimension 1.

By the previous case, there is a one-dimensional $\mathfrak{g}$-submodule L of U complementary to $\mathrm{Ker}(\rho)$. This $\mathfrak{g}$-module has the trivial $\mathfrak{g}$-action, meaning that every element in L is a $\mathfrak{g}$-module homomorphism from $V \to W$. The element of L that maps to Id_W then has as kernel a $\mathfrak{g}$-module complementary to W. $\square$

3.1 Completion of the Proof of Theorem 1.1

Corollary 3.2. *Let $\mathfrak{g}$ be afinite dimensional, complex Lie algebra. If the Killing form B for $\mathfrak{g}$ is non-degnerate, then $\mathfrak{g}$ is semi-simple.*

Proof. By Theorem 3.1 as a $\mathfrak{g}$-module under the adjoint action $\mathfrak{g}$ decomposes as a direct sun of simple $\mathfrak{g}$-modules:

$$\mathfrak{g} = \oplus_{i \in I} V_i.$$

Since each V_i is a $\mathfrak{g}$-module and hence an ideal of $\mathfrak{g}$ and consequently a subalgebra. Similarly. for $i \neq j$, we have $[V_i, V_j] = 0$, so that the V_i are commuting subalgebras. Since $ad(X)ad(Y) = 0$ if X and Y lie in different simple factors, the bilinear form B_V is an orthogonal direct sum of the forms B_{V_i} and in particular in B_{V_i} is non-degenerater..

Lastly, since V_i is a simple $\mathfrak{g}$-module and $[V_j, V_i] = 0$ for $j \neq i$, it follows that V_i is a simple V_i-module. Thus, it has no non-trivial V_i-submodules, and hence each V_i is either a simple algebra or of dimension 1. But if V_i is of dimension 1, then B_{V_i} is the zero bilinear form. This contradicts the noon-degeneracy of B_{V_i}.

Thus, $\mathfrak{g} = \oplus V_i$ is a decomposition of $\mathfrak{g}$ as a direct sum of commuting simple Lie subalgebras, each of which is an ideal. This proves that $\mathfrak{g}$ is a semi-simple algebra. □

This completes the proof of Theorem 1.1.

4 Texchnical Results

Here we establish the technical results we needed in the study of semi-simple algebras. about semi-simple Lie algebras:: (i) the existence of a Cartan subalgebra, (ii) that a Cartan is its own centralizer. In addition, we establish that for any root α there is a subalgebra, which we denote $\mathfrak{s}l(2)_\alpha$ isomorphic to $\mathfrak{s}l(2, \mathbb{C})$ and containing $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ as the root spaces of this semi-simple subalgebra.

4.1 x_{ss} and x_n for x in a semi-simple Lie algebra

Let $x \in \mathfrak{g}l(V)$ (with V a finite dimensional complex vector space). Then the Jordan-Holder decomposition is $x = x_{ss} + x_n$ where x_{ss} is semi-simple and x_n is nilpotent and $[x_{ss}, x_n] = 0$. There is a unique such decomposition.

Claim 4.1. *Consider $ad(x) \in \mathfrak{g}l(\mathfrak{g}l(V))$. Its Jordan-Holder decomposition is*

$$ad(x) = ad(x_{ss}) + ad(x_n),$$

i.e. as elements in $\mathfrak{g}l(\mathfrak{g}l(V))$ we have $(ad(x))_{ss} = ad(x_{ss})$ and $(ad(x))_n = ad(x_n)$.

Proof. $ad(x) = ad(x_{ss}) + ad(x_n)$ and $[ad(x_{ss}), ad(x_n)] = ad([x_{ss}, x_n]) = 0$. Since x_{ss} is semi-simple, choosing a basis in which it is diagonal, we see that $ad(x_{ss})$ is semi-simple with invariant spaces being the one-dimensional subspaces $\{E_{i,j}\}_{1\leq i,j\leq n}$ with non-zero entry only in the $(i,j)^{th}$-position. As we have seen before, the fact that x_n is nilpotent implies that $ad(x_n)$ is nilpotent. Since the Jordan Holder decomposition of $ad(x)$ is unique, this completes the proof. □

We proved in Lemma 1.1 of Lecture VI that for $x \in \mathfrak{gl}(V)$ there are polynomials $p(T)$ and $q(T)$ with zero constant term such that $x_{ss} = p(x)$ and $x_n = q(x)$. Applying this to $ad(x)$ yields the following.

Corollary 4.2. *Given $x \in \mathfrak{gl}(V)$ there are polynomials $p(T)$ and $q(T)$with zero constant term such that $ad(x_{ss}) = p(ad(x))$ and $ad(x_n) = q(ad(x))$.*

Theorem 4.3. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semi-simple subalgebra and let $x \in \mathfrak{g}$. Let $x = x_{ss} + x_n$ be the Jordan-Holder decomposition of $x \in \mathfrak{gl}(V)$ into commuting semi-simple and nilpotent elements of $\mathfrak{gl}(V)$. Then x_{ss} and x_n are contained in $\mathfrak{g}$.*

Proof.

Claim 4.4. *Consider the sub Lie algebra of $\mathfrak{gl}(V)$ consisting of all $X \in \mathfrak{gl}(V)$ with the following properties:*

- *For each $\mathfrak{g}$-submodule $W \subset V$, $X \cdot W \subset W$ and $Tr_W(X) = 0$.*
- *$[X, \mathfrak{g}] \subset \mathfrak{g}$.*

This subset is a sub Lie algebra $\mathfrak{g}' \subset \mathfrak{gl}(V)$ that contains $\mathfrak{g}$ as an ideal.

Proof. (of claim) Fixing a $\mathfrak{g}$-submodule W, the set of X that stabilize W is a sub Lie algebra of $\mathfrak{gl}(V)$, and since $Tr([X,Y]) = 0$ for all X, Y, the X that stabilize W and have trace 0 when restricted to W is a sub Lie algebra. Also the set of X with $[X, \mathfrak{g}] \subset \mathfrak{g}$ is a sub Lie algebra. Hence, the intersection of all these sub Lie algebras is a sub Lie algebra. Since $\mathfrak{g}$ is semi-simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, thus the trace of $X \in \mathfrak{g}$ on any $\mathfrak{g}$-module is trivial. From this it follows immediately that $\mathfrak{g} \subset \mathfrak{g}'$. Since $[\mathfrak{g}', \mathfrak{g}] \subset \mathfrak{g}$, we have that $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$. □

Now let us turn to the proof of the theorem. Let $\mathfrak{g}'$ be as in the claim. Since $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$, it follows that $\mathfrak{g}'$ is a $\mathfrak{g}$-module. By Theorem 3.1, there is a $\mathfrak{g}$-submodule $U \subset \mathfrak{g}'$ with $\mathfrak{g}' = \mathfrak{g} \oplus U$. Since U is a $\mathfrak{g}$-submodule

$[\mathfrak{g}, U] \subset U$. But since $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$, $[U, \mathfrak{g}] \subset \mathfrak{g}$. This implies that $[\mathfrak{g}, U] = 0$, i.e., that all $X \in U$ commute with $\mathfrak{g}$. This means that for every $X \in U$ the linear map $X : V \to V$ is a $\mathfrak{g}$-module homomorphism. Since $U \subset \mathfrak{g}'$, for $W \subset V$ a simple $\mathfrak{g}$-submodule X maps W to W and its restriction to W has zero trace. Since X is a $\mathfrak{g}$-module map and W is a simple $\mathfrak{g}$-module $X|_W$ is multiplication by a constant. Since the trace of this map is zero, that constant is 0. Hence, the restriction of X to any simple $\mathfrak{g}$-submodule of V is trivial. On the other hand, by complete reduciblility of $\mathfrak{g}$-modules, the $\mathfrak{g}$-module V is a direct sum of simple $\mathfrak{g}$-modules and X acts trivially on each of these summands. Hence, $X = 0$. Since X was an arbitrary element of U, this shows that $U = 0$ and hence $\mathfrak{g} = \mathfrak{g}'$.

Now we have seen (Lemma 1.1 in Lecture VI) that both x_{ss} and x_n are given by polynomials in x with zero constant term. Fix a $\mathfrak{g}$-submodule W of V. Since $x \in \mathfrak{g}$, it follows that $x \cdot W \subset W$. Thus, for any polynomial $p(x)$ we have $p(x) \cdot W \subset W$. This shows that $x_{ss} \cdot W \subset W$ and $x_n \cdot W \subset W$. Since x_n is nilpotent, its restriction to W is nilpotent and hence has zero trace. Since $x_{ss} = x - x_n$ and both x and x_n have zero trace on W, the same is true of s_{ss}. Lastly, we must show that $[x_{ss}, \mathfrak{g}] \subset \mathfrak{g}$, and similarly for x_n. But $ad(x_{ss})$ is a polynomial in $ad(x)$ with zero constant term since $ad(x)(\mathfrak{g}) \subset \mathfrak{g}$, we also have $ad(x_{ss})(\mathfrak{g}) \subset \mathfrak{g}$. An analogous argument shows $ad(x_n)(\mathfrak{g}) \subset \mathfrak{g}$. This shows that x_{ss} and x_n are elements of $\mathfrak{g}' = \mathfrak{g}$, completing the proof. □

Definition 4.5. Since the adjoint representation, $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, is injective, for any $x \in \mathfrak{g}$ it is natural to define x_{ss} and x_n in $\mathfrak{g}$ so that $ad(x_{ss}) = ad(x)_{ss}$ and $ad(x_n) = ad(x)_n$. Since ad is an injection and $ad(x)_{ss}$ and $ad(x)_n$ are both contained in the image of the adjoint representation, this does indeed produce elements x_{ss} and x_n in $\mathfrak{g}$.

Lemma 4.6. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and fix $x \in \mathfrak{g}$. Suppose $x = x_{ss} + x_n$ is as in Definition 4.5. Then x_{ss} is ad-semi-simple, x_n is ad nilpotent, and $[x_{ss}, x_n] = 0$. Furthermore, x_{ss} and x_n are the unique pair of elements summing to x with these three properties.*

Proof. This is obvious from the definition, the fact that the adjoint representation is injective, and the uniqueness of the Jordan-Holder decomposition for any element in the endomorphism algebra of a finite dimensional complex vector space. □

Now let us show that the decomposition $x = x_{ss} + x_n$ continues to hold under the image of any representation of $\mathfrak{g}$.

Proposition 4.7. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and let $x \in \mathfrak{g}$. Suppose that $\rho \colon \mathfrak{g} \otimes V \to V$ is a finite dimensional $\mathfrak{g}$-module. Then $(\rho(x))_{ss} = \rho(x_{ss})$ and $(\rho(x))_n = \rho(x_n)$*

Proof. Let $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into its simple (commuting) factors. Suppose $x \in \mathfrak{g}$. We write $x = \sum_i x_i$ with $x_i \in \mathfrak{g}_i$. Since $ad_{\mathfrak{g}} = \sum_i ad_{\mathfrak{g}_i}$, we see that $x_{ss} = \sum_i (x_i)_{ss}$ and $x_n = \sum_i (x_i)_n$. Thus, it suffices to prove the proposition when $\mathfrak{g}$ is a simple Lie algebra. Assuming that $\mathfrak{g}$ is simple, any $\rho \colon \mathfrak{g} \to \mathfrak{g}l(V)$ is either 0 or an embedding. If ρ is zero the result is obvious.

Now suppose that $\mathfrak{g}$ is simple and ρ is an embedding. According to Proposition 4.3, the elements $(\rho(x))_{ss}$ and $(\rho(x))_n$ in $\mathfrak{g}l(V)$ are contained in $\rho(\mathfrak{g})$. Let x_{ss}^V and x_n^V be the elements in $\mathfrak{g}$ with $\rho(x_{ss}^V) = (\rho(x))_{ss}$ and $\rho(x_n^V) = (\rho(x))_n$. The element $ad(\rho(x_{ss}^V))$ is semi-simple on $\mathfrak{g}l(V)$ and the element $ad(\rho(x_n^V))$ is nilpotent on $\mathfrak{g}l(V)$. Thus, $ad(\rho(x_{ss}^V))$ is semi-simple on $\rho(\mathfrak{g})$ and $ad(\rho(x_n^V))$ is nilpotent on $\rho(\mathfrak{g})$. Since $\rho(x_{ss}^V)$ and $\rho(x_n^V)$ commute, the same is true of $ad(\rho(x_{ss}^V))$ and $ad(\rho(x_n^V))$. This means that $ad(\rho(x_{ss}^V))$ and $ad(\rho(x_n^V))$ are the semi-simple and nilpotent terms in the Jordan-Holder decomposition of $ad(\rho(x))$ acting on $\mathfrak{g}$. This means that $ad(x_{ss}^V)$ and $ad(x_n^V)$ are the semi-simple and nilpotent terms in the Jordan-Holder decompostion of $ad(x)$. By definition, this means $x_{ss}^V = x_{ss}$ and $x_n^V = x_n$. Thus, $\rho(x_{ss}) = (\rho(x))_{ss}$ and $\rho(x_n) = (\rho(x))_n$. □

5 Existence of a Cartan subalgebra

Lemma 5.1. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and suppose that L is a subalgebra with the property that for every $x \in L$ is $ad(x)$ semi-simple. Then L is an abelian subalgebra.*

Proof. Let $x \in L$ and we prove the result by showing that all the eigenvalues of $ad_L(x)$ are zero. Arguing by contradiction, suppose that there is $y \in L$ an eigenvector for $ad_L(x)$ with non-zero eigenvalue a, $ad_L(x)(y) = ay$. Now $ad_L(y)(x)$ is a sum of eigenvectors of $ad_L(y)$ with non-zero eigenvalue. But $ad_L(y)(x) = -ad_L(x)y = -ay$, and this is an eigenvector of $ad_L(y)$ with zero eigenvalue. This is a contradiction, and hence all eigenvalues of $ad_L(x)$ are zero. □

Lemma 5.2. *If $\mathfrak{g}$ is a semi-simple complex Lie algebra, then there is a semi-simple element in $\mathfrak{g}$.*

Proof. If $x \in \mathfrak{g}$ has a Jordan decomposition $x = x_{ss} + x_n$ then both x_{ss} and x_n are contained in $\mathfrak{g}$. Thus, either every element of $\mathfrak{g}$ is nilpotent or there is a semi-simple element in $\mathfrak{g}$. But by Corollary 2.4 of Lecture VIB, if every element of $\mathfrak{g}$ is nilpotent then $\mathfrak{g}$ is a nilpotent Lie algebra. Since every non-zero nilpotent algebra has a non-trivial center, and hence has a one-dimensional ideal. Either this is a proper ideal or the dimension of the Lie algebra is one. In both cases the Lie algebra is not semi-simple. □

Corollary 5.3. *There are subalgebras $\mathfrak{h} \subset \mathfrak{g}$ consisting of only semi-simple elements. A maximal such one is a Cartan subalgebra of $\mathfrak{g}$*

5.1 $\mathfrak{h}$ is its own Centralizer $\mathfrak{g}$

For this subsection we fix a semi-simple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$.

Now we write

$$\mathfrak{g} = C \oplus_\alpha \mathfrak{g}_\alpha$$

where C is the subspace that commutes with $\mathfrak{h}$ (the 0 eigenspace), where α ranges over non-zero elements in $\mathfrak{h}^*$, and where $\mathfrak{g}_\alpha$ is the α-eigenspace for $\mathfrak{h}$.

Claim 5.4. *If $a, b \in \mathfrak{h}^*$ are eigenvalues for the action of $\mathfrak{h}$ on $\mathfrak{g}$ and $a+b \neq 0$, then $B(\mathfrak{g}_a, \mathfrak{g}_b) = 0$*

Proof. Since $ad(\mathfrak{g}_a)(\mathfrak{g}_c) \subset \mathfrak{g}_{a+c}$, the composition $ad(\mathfrak{g}_a) \circ ad(\mathfrak{g}_b)$ sends $\mathfrak{g}_c$ to $\mathfrak{g}_{a+b+c}$ and as long as $a + b \neq 0$ the trace of such a map is zero. □

Corollary 5.5. *The restriction of B to $C \otimes C$ is non-degenerate.*

Proposition 5.6. *$C = \mathfrak{h}$. That is to say $\mathfrak{h}$ is its own centralizer.*

Proof. **Step 1. The semi-simple part of every element of C is contained in $\mathfrak{h}$.**

Let $x \in C$ and we write the Jordan decomposition $x = x_{ss} + x_n$. Since $ad(x_{ss})$ is a polynomial with zero constant term in $ad(x)$ and $\mathfrak{h} \subset \mathrm{Ker}(ad((x)$, it follows that $\mathfrak{h} \in \mathrm{Ker}(ad(x_{ss}))$. The maximality of $\mathfrak{h}$ among subalgebras consisting only of semi-simple elements means that $x_{ss} \in \mathfrak{h}$.

Step 2. The Killing form restricted to $\mathfrak{h}$ is non-degenerate.

Suppose that $h \in \mathfrak{h}$ and $B(h, \mathfrak{h}) = 0$. To establish the statement we show that this implies that $h = 0$. Let $x \in C$ with Jordan decomposition $x = x_{ss} + x_n$. By Step 1 the element $x_{ss} \in \mathfrak{h}$ so that $B(h, x_{ss}) = 0$. Since x_n is nilpotent, by Lemma 1.3 of Lecture VIB, $ad(x_n)$ is nilpotent. Since $[h, x_n] = 0$ the elements $ad(x_n)$ and $ad(h)$ commute. But the composition of commuting elements, one of which is nilpotent, is nilpotent. This implies that the composition $ad(h)ad(x_n)$ is nilpotent and hence has zero trace. This shows that $B(h, C) = 0$. By Corollary 5.5 the restriction of B to $C \otimes C$ is non-degenerate, so $h = 0$, proving that the restriction of B to $\mathfrak{h} \otimes \mathfrak{h}$ is non-degenerate.

Step 3. C is nilpotent.

For any $x \in C$ with $x = x_{ss} + x_n$ as above, we know that $x_{ss} \in \mathfrak{h}$ and thus $ad_C(x_{ss}) = 0$. On the other hand, by Lemma 1.3 of Lecture VIB, since x_n is nilpotent $ad_C(x_n)$ is also nilpotent. This shows that $ad_C(x)$ is nilpotent for every $x \in C$. From this it follows by Corollary 2.4 of Lecture VIB that C is a nilpotent Lie algebra.

Step 4. $\mathfrak{h} \cap [C, C] = 0$.

For $h \in \mathfrak{h}$ and $x, y \in C$ we have $B(h, [x, y]) = B([h, x], y) = B(0, y) = 0$. Since the restriction of B to $\mathfrak{h} \otimes \mathfrak{h}$ is non-degenerate the statement follows.

Step 5. $[C, C] = 0$.

Claim 5.7. *If N is a nilpotent Lie algebra and if $I \subset N$ is a non-zero ideal, then the intersection of I and the center of N is non-zero.*

Proof. I is an N-module via the adjoint representation. According to Proposition 2.3 of Lecture VIB for every $x \in N$ the adjoint representation $ad(x)$ is nilpotent. Thus, the action of N on I consists of nilpotent transformations. Therefore by Lemma 1.4 of Lecture VJB, there is a flag $0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = I$ of N submodules of I with the property that $[N, I_j] \subset I_{j-1}$. In particular, $[N, I_1] = 0$ and hence I_1 is contained in the center of N. $\square$

Since we know that C is nilpotent, if $[C, C] \neq 0$ then by the claim there is $0 \neq z \in [C, C]$ that is in the center of C. It cannot be the case that $z \in \mathfrak{h}$ since we have already seen that $\mathfrak{h} \cap [C, C] = 0$. Thus, $z = z_{ss} + z_n$ and since $z_{ss} \in \mathfrak{h}$, we have $z_n \neq 0$. Since $z_{ss} \in \mathfrak{h}$ is contained in the center of C, we have that z_n is contained in the center of C. By Lemma 1.3 of Lecture VIB,, since z_n is nilpotent, $ad_{\mathfrak{g}}(z_n)$ is nilpotent. Since z_n commutes with every $c \in C$, $ad_{\mathfrak{g}}(z_n)$ commutes with $ad_{\mathfrak{g}}(c)$ for every $c \in C$. But the composition of commuting elements, one of which is nilpotent is nilpotent, so that $ad_{\mathfrak{g}}(z_n) \circ ad_{\mathfrak{g}}(c)$ is nilpotent. Thus, $B(z_n, c) = Tr(ad_{\mathfrak{g}}(z_n) \circ ad_{\mathfrak{g}}(c)) = 0$. Since this is true for all $c \in C$ and since by Corollary 5.5 the restriction of the Killing form to $C \otimes C$ is non-degenerate, this implies that $z_n = 0$. This proves that $[C, C] = 0$.

Step 6. $C = \mathfrak{h}$.

If $C \neq \mathfrak{h}$ let $z \in C \setminus \mathfrak{h}$ and as before write $z = z_{ss} + z_n$ with $z_{ss} \in \mathfrak{h}$. Thus, $z_n \in C \setminus \mathfrak{h}$. Since C is abelian $ad_{\mathfrak{g}}(z_n)$ commutes with $ad_{\mathfrak{g}}(c)$ for every

$c \in C$. But $ad_{\mathfrak{g}}(z_n)$ is nilpotent, thus so is $ad_{\mathfrak{g}}(z_n) \circ ad_{\mathfrak{g}}(c)$. This shows that $B(z_n, c) = 0$ for all $c \in C$. By the non-degeneracy of the restriction of B to $C \otimes C$, we conclude that $z_n = 0$, which is a contradiction. □

5.2 Summary

Now we have established the basic facts: (i) existence of a Cartan subalgebra, (ii) the fact that a Cartan is it is its own centralizer, and (iii) the Killing form is non-degenerate. The one other result result we used in Lecture VI was that any element of the derived sub algebra of a solvable Lie algebra $\mathfrak{g}$ acts in a nilpotent way on any $\mathfrak{g}$-module. This follows from the facts that (i) the nilradical of a solvable algebra is its derived subalgebra (Theorem 3.3 of Lecture VIB), (ii) that the nilradical of a Lie algebra $\mathfrak{g}$ is contained in the nilpotent ideal of any $\mathfrak{g}$-module (Theorem 2.12 of Lecture VIB), and (iii) that every element in the nilpotent ideal of a $\mathfrak{g}$-module V acts by a nilpotent transformation on V (Corollary 2.8 and Definition 2.9 of Lecture VIB).

Here we recap what we have now established. Fix a semi-simple Lie algebra $\mathfrak{g}$.

(i) $\mathfrak{g}$ has a *Cartan subalgebra*; i.e., a maximal abelian subalgebra all of whose elements are semi-simple.

(ii) The centralizer of any Cartan subalgebra $\mathfrak{h}$ is itself.

(iii) $\mathfrak{g}$ is an $\mathfrak{h}$-module by the restriction to $\mathfrak{h}$ of the adjoint representation. This module decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where Φ is a finite subset of $\mathfrak{h}^* \setminus \{0\}$. The $\alpha \in \Phi$ are the *roots* and the $\mathfrak{g}_\alpha$ are the *root spaces*.

(iv) The root spaces $\mathfrak{g}_\alpha$ are one-dimensional.

(v) The bracket $[\mathfrak{h}, \mathfrak{g}_\alpha]$ is given by $[h, X] = \alpha(h)X$.

(vi) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(vii) The restriction of the Killing form B to $\mathfrak{h}$ is non-degenerate and hence identifies $\mathfrak{h}$ and $\mathfrak{h}^*$.

(viii) For each root α, setting $t_\alpha \in \mathfrak{h}$ equal to the element corresponding to $\alpha \in \mathfrak{h}^*$ under the isomorphism in (vii) we have $\alpha(t_\alpha) \neq 0$.

(ix) The subalgebra $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathbb{C}(t_\alpha)$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. It is denoted $\mathfrak{sl}(2)_\alpha$

(x) The Killing form B induces a non-degenerate form on $\mathfrak{h}^*$ which we denote by $\langle a, b\rangle$. For any root α we have

$$B(t_\alpha, t_\alpha) = \alpha(t_\alpha) = \langle \alpha, \alpha\rangle.$$

(xii) For a root α we define

$$h_\alpha = \frac{2t_\alpha}{B(t_\alpha, t_\alpha)} = \frac{2t_\alpha}{\langle \alpha, \alpha\rangle}.$$

Fixing $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ with $B(X, Y) = 2/B(t_\alpha.t_\alpha)$, the three elements X, h_α, Y correspond under an isomorphism of $\mathfrak{sl}(2)_\alpha$ with $\mathfrak{sl}(2, \mathbb{C})$ with the standard generators

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(xiii) For roots α and β we define the *Cartan integer* $n(\alpha, \beta)$ by

$$n(\alpha, \beta) = \frac{2\alpha(h_\beta)}{\alpha(h_\alpha)} = \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}.$$

Here are the properties of the root system of a semi-simple Lie algebra:

(a) The roots span $\mathfrak{h}^*$.

(b) The Cartan integers are indeed integers and for every root $n(\alpha, \alpha) = 2$.

(c) Let $\mathfrak{h}^*_\mathbb{Q} \subset \mathfrak{h}^*$ be the rational subspace generated by all the roots. Then $\mathfrak{h}^*_\mathbb{Q}$ is a rational form of $\mathfrak{h}^*$ in the sense that $\mathfrak{h}^*_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C} = \mathfrak{h}^*$.

(d) The restriction of the dual to the Killing form to $\mathfrak{h}^*_\mathbb{Q}$ is a positive definite, rational bilinear form

(e) The rational dual of $\mathfrak{h}^*_\mathbb{Q}$ is the rational subspace of $\mathfrak{h}$ generated by the $\{h_\alpha\}_{\alpha\in\Phi}$.

(f) If α is a root then $-\alpha$ is a root and $\pm\alpha$ are the only multiples of α that are roots.

(g) For each root α, there is an involution I_α of $\mathfrak{h}$ that preserves the Killing form, sends h_α to $-h_\alpha$.

(h) The formula for I_α is:

$$I_\alpha(h) = h - \frac{2\alpha(h)}{\langle \alpha, \alpha \rangle} h_\alpha.$$

The dual involution $I_\alpha^* \colon \mathfrak{h}^* \to \mathfrak{h}^*$ is given by

$$I_\alpha^*(a) = a - \frac{2\langle a, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha,$$

thus on a root β is given by the formula

$$I_\alpha^*(\beta) = \beta - n(\alpha, \beta)\alpha.$$

The involution I_α^* preserves the set of roots.

6 Axioms for Root Systmes

Definition 6.1. More generally, a *root system* consists of a triple (V, B, Φ) where V is a finite dimensional real vector space, B is a positive definite bilinear pairing on V, denoted $\langle \cdot, \cdot \rangle$, and Φ is a finite subset of V, the set of *roots*. These data are required to satisfy the following properties:

- The roots span V.
- If α is a root, then so is $-\alpha$ but no other multiple of α is a root.
- For α, β roots, the quantity $n(\alpha, \beta) = 2\langle \alpha, \beta \rangle / \langle \alpha, \beta \rangle$ is an integer.
- For each root α, there is an involution $I_\alpha \colon V \to V$ defined by

$$I_\alpha(v) = v - \frac{2\langle \alpha, v \rangle}{\langle \alpha, \alpha \rangle} \alpha.$$

  This involution preserves the set Φ and the blinear form B.