Lie Groups: Fall, 2022
Lecture VIB: Nilpotent and Solvable Lie Algebras

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In order to prepare the way to establish the various technical results we used in our study of semi-simple Lie algebras, we study nilpotent and solvable Lie algebras.

1 Nilpotency

Definition 1.1. By a *flag* in a finite dimensional vector space V we mean an increasing sequence of subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$

where for each $1 \leq i \leq k$ the inclusion $V_{i-1} \subset V_i$ is a proper inclusion.

1.1 Nilpotent Elements

Definition 1.2. Recall that an element $x \in \mathfrak{g}l(V)$ is *nilpotent*, or more precisely is a *nilpotent transformation* of V if $0 = x^n \in \mathfrak{g}l(V)$ for some $n > 0$.

Lemma 1.3. *Let V be a finite dimensional complex vector space. If $x \in \mathfrak{g}l(V)$ is nilpotent, then $ad(x) \in \mathfrak{g}l(\mathfrak{g}l(V))$ is also nilpotent trransformation of $\mathfrak{g}l(V)$.*

Proof. $ad(x)^k(y)$ is a sum of terms of the form $\pm x^i y x^j$ where $i + j = k$. $\square$

Lemma 1.4. *Let $V \neq \{0\}$ be a finite dimensional complex vector space. If $\mathfrak{g} \subset \mathfrak{g}l(V)$ is a sub Lie algebra and if every element of $\mathfrak{g}$ is a nilpotent transformation of V, then there is a flag*

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$

with the property that $\mathfrak{g} \cdot V_i \subset V_{i-1}$ for all $i \geq 1$.

Proof. The proof is by induction on the dimension of $\mathfrak{g}$. If the dimension of $\mathfrak{g} = 0$, then there is nothing to prove. Suppose that the dimension of $\mathfrak{g}$ is $k > 0$ and the result holds for all Lie algebras of dimension $< k$.

For any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of dimension less than $\mathfrak{g}$, consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Since every element of $\mathfrak{h}$ is a nilpotent transformation of V, it follows from Lemma 2.3 that this action consists of nilpotent transformations of $\mathfrak{g}$. Clearly, this action preserves $\mathfrak{h}$ and hence there is an induced action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, every element of which is nilpotent. By induction, the subspace $W \subset \mathfrak{g}/\mathfrak{h}$ annihilated by $\mathfrak{h}$ is non-zero. Fix $y \neq 0$ in W. Then $[\mathfrak{h}, y] \subset \mathfrak{h}$. That is to say $\mathfrak{h}' = \mathfrak{h} \oplus \mathbb{C}(y)$ is a lie subalgebra and $\mathfrak{h} \subset \mathfrak{h}'$ is an ideal of codimension 1. Arguing by induction on dimension (starting with $\mathfrak{h} = 0$), we see that there is an ideal I of codimension 1 in $\mathfrak{g}$.

By the inductive hypothesis, the subspace, $W \subset V$, of V annihilated by I is non-zero. Let $y \in \mathfrak{g} \setminus I$. Since y normalizes I, it preserves W. Since y is a nilpotent transformation of V its restriction to W is a nilpotent transformation. Hence, there is a non-zero vector $w \in W$ in the kernel of y. Since I and y together generate $\mathfrak{g}$, $\mathfrak{g}$ annihilates w.

This produces $0 \subset V_1 \subset V$ with $V_1 \neq 0$ and V_1 in the kernel of $\mathfrak{g}$. Consider the quotient $W = V/V_1$. If $W = \{0\}$, then $0 = V_0 \subset V_1 = V$ is the required flag. Otherwise consider the induced action of $\mathfrak{g}$ on W. It is an action by nilpotent elements. Hence, by induction on the dimension of $\mathfrak{g}$-modules, we can assume that there is a flag $0 \subset W_1 \subset W_2 \cdots \subset W_k = W$ as stated in the lemma. For $i \geq 2$ set V_i equal to the preimage of W_{i-1} under the natural projection $V \to W$. Then

$$0 \subset V_1 \subset V_2 \cdots \subset V_{k+1} = V$$

is the required flag in V. □

2 Nilpotent Lie Algebras

Definition 2.1. For any Lie algebra $\mathfrak{g}$ we define ideals inductively by setting $\mathfrak{g}_1 = \mathfrak{g}$ and $\mathfrak{g}_m = [\mathfrak{g}, \mathfrak{g}_{m-1}]$. This is the *lower central series* for $\mathfrak{g}$. Notice that $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$ is the derived subalgebra of $\mathfrak{g}$.

A Lie algebra is *nilpotent* if the lower central series terminates in 0 at some finite stage. That is to say, if $\mathfrak{g}_m = 0$ for some m.

Lemma 2.2. *If $\mathfrak{g}$ is a non-zero nilpotent Lie algebra then its center is non-zero*

Proof. The last non-zero term in the lower central series is contained in the center of the Lie algebra. □

Proposition 2.3. *The following are equivalent:*

- $\mathfrak{g}$ *is a nilpotent Lie algebra.*
- *For some* $k > 0$ *all elementary brakcets* $[x_1, [x_2, [\cdots [x_{k-1}, x_k]] \cdots]$ *vanish.*
- *For some* $k > 0$ *for all* $x_1, \ldots, x_k \in \mathfrak{g}$ *we have*

$$ad(x_1)ad(x_2)\cdots ad(x_k) = 0.$$

Proof. Since

$$ad(x_1)ad(x_2)\cdots ad(x_{k-1})(x_k) = [x_1, [x_2, \cdots [x_{k-1}, x_k]] \cdots]$$

the second and third item are equivalent. On the other hand, $\mathfrak{g}_m$ is the vector space spanned by the m-fold brackets as in the second item. This shows that the first and second items are equivalent. □

Corollary 2.4. *If* $\mathfrak{g} \subset \mathfrak{g}l(V)$ *is a Lie algebra consisting of elements that are nilpotent endomorphisms of* V*, then* $\mathfrak{g}$ *is a nilpotent Lie Algebra.*

Proof. According to Lemma 2.4 there is a flag in $0 \subset V_0 \subset V_1 \subset \cdots \subset V_k = V$ such that for each i we have $\mathfrak{g} \cdot V_i \subset V_{i-1}$. It follows that for each i the subalgebra $\mathfrak{g}_r$ maps V_i to V_{i-r}. Hence, for r sufficiently large $\mathfrak{g}_r = 0$. □

Notice that the converse to this corollary is not true. A subalgebra $\mathfrak{g}$ of $\mathfrak{g}l(V)$ can be a nilpotent Lie algebra without consisting of nilpotent elements. An example is subalgebra of diagonal matrices. It is an abelian subalgebra hence nilpotent but its elements are semi-simple.

Corollary 2.5. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra. Then* $\mathfrak{g}$ *is nilpotent if and only if* $ad(x)$ *is nilpotent for every* $x \in \mathfrak{g}$.

Proof. If $ad(x)$ is nilpotent for every $x \in \mathfrak{g}$, then by Lemma 2.4 applied to the adjoint representation of $\mathfrak{g}$ there is a flag

$$0 \subset V_1 \subset V_2 \cdots \subset V_k = \mathfrak{g}$$

with $[\mathfrak{g}, V_i] \subset V_{i-1}$. It now follows immediately from Proposition 3.3 that $\mathfrak{g}$ is nilpotent.

Conversely, if $\mathfrak{g}$ is nilpotent then the lower central series produces a flag as above so that $ad(x)$ is nilpotent for every $x \in \mathfrak{g}$. □

2.1 Jordan-Holder Series and Nilpotence

Definition 2.6. Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}$-module V is *simple* if V has no non-trivial submodules.

Let $M \neq \{0\}$ be a finite dimensional $\mathfrak{g}$ module. A *Jordan-Holder series* for M is a sequence of $\mathfrak{g}$-submodules

$$M = M_0 \supset M_1 \supset M_2 \cdots \supset M_k = 0,$$

where for each i the quotient M_i/M_{i+1} is a simple $\mathfrak{g}$-module.

It is easy to see that there always exists a Jordan-Holder series (which is not necessarily unique) for any (non-zero) finite dimensional $\mathfrak{g}$-module.

Lemma 2.7. *Let M be a finite dimensional $\mathfrak{g}$-module and let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ every element of which is a nilpotent endomorphism of M. Let $\{M_i\}$ be a Jordan-Holder decomposition for M. Then for every i the map induced by any $x \in \mathfrak{a}$ on M_i/M_{i+1} is zero. In particular, if M is simple, then $\mathfrak{a}$ acts trivially on M.*

Proof. For each i we have an induced action of $\mathfrak{a}$ on M_i/M_{i+1} and every element of $\mathfrak{a}$ acts by a nilpotent transformation on this sub-quotient. Thus, by Lemma 2.4, the subspace $W \subset (M_i/M_{i+1})$ annihilated by every $x \in \mathfrak{a}$ is non-zero. Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ the subspace W is stable under $\mathfrak{g}$. Hence, W is a non-zero $\mathfrak{g}$ submodule of M_i/M_{i+1}. Since M_i/M_{i+1} is simple, $W = (M_i/M_{i+1})$, showing that the action of $\mathfrak{a}$ on M_i/M_{i+1} is trivial. $\square$

Corollary 2.8. *Let M be a finite dimensional $\mathfrak{g}$-module and $\{M_i\}$ a Jordan-Holder sequence for M. Then the set of $x \in \mathfrak{g}$ such that for each i the map induced by x from M_i/M_{i+1} to itself is zero is an ideal. This ideal consists of elements of $\mathfrak{g}$ whose action sends each M_i to M_{i+1} and hence are nilpotent transformations of M.*

Proof. Let $\mathfrak{n}$ be the set of $x \in \mathfrak{g}$ with the property that $x \cdot M_i \subset M_{i+1}$ for every i. Suppose $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$. Since $y \cdot M_i \subset M_i$, it follows that $[y, x] \in \mathfrak{n}$. This proves that $\mathfrak{n}$ is an ideal. It is clear that every element of $\mathfrak{n}$ acts nilpotently on M. $\square$

Definition 2.9. This ideal in the previous corollary is denoted $\mathfrak{n} = \mathfrak{n}(M)$ and is the *nilpotent ideal* of the $\mathfrak{g}$-module M.

2.2 The (Jacobson) Radical of an Algebra

Recall that if A is an associative algebra with unit, then the *(Jacobson) radical* $J(A)$ of A is the intersection of all maximal left ideals. This is the same as the two-sided ideal of all $a \in A$ such that $a \cdot M = 0$ for every simple left A-module M.

Lemma 2.10. *Suppose that A is a finite dimensional $\mathbb{C}$-algebra. Then $J(A)$ is a nilpotent ideal in the sense that for some $k > 0$ we have $(J(A))^k = 0$.*

Proof. $J(A)$ is the intersection of two-sided ideals and hence is a two-sided ideal of A. By the hypothesis on A, the ideal $J(A)$ is finite dimensional over $\mathbb{C}$. Thus, $J(A)$ has a Jordan-Holder decomposition $\{J_i\}$ with $J_i \supset J_{i+1}$ and the quotients J_i/J_{i+1} being simple A-modules. From the definition of $J(A)$, we see that its action on J_i/J_{i+1} is trivial for all i. Thus, $J(A) \otimes J_i(A) \to J_{i+1}(A)$, and consequently $J(A)$ is nilpotent in the sense that $J(A)^k = 0$ for some $k > 0$. □

2.3 The Nilradical of $\mathfrak{g}$

Definition 2.11. The *nilradical* $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the intersection of the kernels of all simple representations of $\mathfrak{g}$. Said another way, it consists of all $x \in \mathfrak{g}$ such that $x \cdot M = 0$ for every simple $\mathfrak{g}$-module M.

Theorem 2.12. *The nilradical of $\mathfrak{g}$ is a nilpotent ideal of $\mathfrak{g}$. It is contained in the nilpotent ideal of any non-zero, finite dimensional $\mathfrak{g}$-module.*

Proof. Since $\mathcal{N}$ is an intersection of ideals of $\mathfrak{g}$, it is an ideal. To prove that the nilradical is a nilpotent ideal we show that $ad(x)$ is a nilpotent transformation of $\mathfrak{g}$ for every $x \in \mathcal{N}$ and then invoke Corollary 3.5. To do this let A be the associative subalgebra (over $\mathbb{C}$) of $\mathfrak{gl}(\mathfrak{g})$ generated by 1 and $ad(x)$ for all $x \in \mathfrak{g}$. Then A is a finite dimensional algebra over $\mathbb{C}$. By definition $\mathfrak{g} \subset A$ and the Lie bracket on $\mathfrak{g}$ is induced by the $ab - ba$ product in the associative algebra A. Thus, any A-module is a $\mathfrak{g}$-module and since $\mathfrak{g}$ generates A as an algebra, any simple A-module is a simple $\mathfrak{g}$-module.

Since every element in $\mathcal{N}$ acts trivially on every simple $\mathfrak{g}$-module, *a fortiori* it acts trivially on every simple A-module. This shows that $\mathcal{N}$ is contained in the radical $J(A)$. But as we have seen $J(A)^k = 0$ for some $k > 0$. Thus, for every $x \in \mathcal{N}$ the action $ad(x)$ on $\mathfrak{g}$ is nilpotent. By Corollary 3.5 this implies that $\mathcal{N}$ is a nilpotent ideal of $\mathfrak{g}$.

Lastly, suppose that $M \neq \{0\}$ is a finite dimensional $\mathfrak{g}$ module. Let M_i be its Jordan-Holder composition series. For each i, the module M_i/M_{i+1} is a simple $\mathfrak{g}$-module and hence the nilradical acts trivially onM_i/M_{i+1}. That is to say the nilradical maps M_i to M_{i+1}. According to Definition 3.9, this means that the nilradical of $\mathfrak{g}$ is contained in the nilpotent ideal of M. $\square$

3 Solvable Lie Algebras and the Radical of any Lie Algebra

3.1 Solvable Lie Algebras

Definition 3.1. Given a Lie algebra $\mathfrak{g}$ we define the *derived series* inductively by $\mathcal{D}_1(\mathfrak{g}) = \mathfrak{g}$ and for $m > 1$:

$$\mathcal{D}_m(\mathfrak{g}) = [\mathcal{D}_{m-1}(\mathfrak{g}), \mathcal{D}_{m-1}(\mathfrak{g})].$$

A Lie algebra is *solvable* if $\mathcal{D}_k(\mathfrak{g}) = 0$ for some $k \geq 1$.

Notice $\mathcal{D}_2(\mathfrak{g}) = \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$ is the derived subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}_m \supset \mathcal{D}_m$ any nilpotent algebra is solvable. Also, notice that for every $m \geq 1$ the quotient $\mathcal{D}_m/\mathcal{D}_{m+1}$ is an abelian Lie algebra.

Exercise. Let L be a Lie algebra with an ideal M with quotient the Lie algebra P. Show that L is solvable if and only if M and P are.

3.2 The Radical

It follows from the exercise in the previous subsection that if I and J are solvable ideals of a lie algebra L then so is $I + J$. [J is an ideal of $I + J$ with quotient $I/I \cap J$.] It follows from this that any (finite dimensional) complex lie algebra $\mathfrak{g}$ has a maximal solvable ideal. This ideal contains all other solvable ideals.

Definition 3.2. The *radical* of $\mathfrak{g}$, denoted $\mathfrak{r} = \mathfrak{r}(\mathfrak{g})$, is the maximal solvable ideal of $\mathfrak{g}$.

Exercise. Show that $\mathfrak{g}/\mathfrak{r}$ has trivial radical and that $\mathfrak{r}$ is contained in any ideal with this property.

Here is the fundamental result that gives the relationshp between the radical and nilradical of a Lie algebra.

Theorem 3.3. *For any finite dimensional complex Lie algebra $\mathfrak{g}$ the nilradical of $\mathfrak{g}$ is the intersection of its radical and its derived subgroup:*

$$\mathcal{N}(\mathfrak{g}) = \mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r}.$$

Corollary 3.4. *If L is a solvable Lie algebra, then $\mathcal{D}_2(L)$ is the nilradical of L and hence $\mathcal{D}_2(L)$ is a nilpotent subalgebra.*

Corollary 3.5. *L is solvable if and only if $[L, L]$ is nilpotent.*

Proof. The previous corollary establishes the forward implication. Conversely, since $L/[L, L]$ is abelian and hence solvable, if $[L, L]$ is nilpotent and hence solvable, then it follows that L, which is the extension of the former by the latter, is solvable, □

Corollary 3.6. *The only simple modules of a solvable Lie algebra $\mathfrak{g}$ are one-dimensional and are given by linear maps $\lambda\colon \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathbb{C}$.*

Proof. (of corollary, assuming the theorem) Suppose that $\mathfrak{g}$ is solvable. Then its nilradical is $\mathcal{D}_2(\mathfrak{g})$ and hence every simple $\mathfrak{g}$-module is induced by pulling back a simple $A(\mathfrak{g}) = (\mathfrak{g}/\mathcal{D}_2(\mathfrak{g}))$-module. But $A(\mathfrak{g})$ is an abelian Lie algebra and consequently its only simple modules are one -dimensional and, up to isomorphism, are given by a linear map $\lambda\colon A(\mathfrak{g}) \to \mathbb{C}$. □

Proof. (of theorem) $T = \mathfrak{g}/\mathcal{D}_2(\mathfrak{g})$ is an abelian Lie algebra. Any $\lambda \in T^*$ determines a one-dimensional simple $\mathfrak{g}$ module, which is annihilatd by $\mathcal{N}$. This shows that $\mathcal{N} \subset \mathrm{Ker}(\lambda)$. Since this is true for every $\lambda \in T^*$, $\mathcal{N}$ is contained in the kernel of the quotient map $\mathfrak{g} \to T$ and thus $\mathcal{N} \subset \mathcal{D}_2(\mathfrak{g})$. Since $\mathcal{N}$ is nilpotent, it is solvable and hence contained in $\mathfrak{r}$. This proves that $\mathcal{N} \subset \mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r}$.

To prove the converse we show that for any simple $\mathfrak{g}$-module V the ideal $\mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r}$ annihilates V. We fix a simple $\mathfrak{g}$-module V and $\rho\colon \mathfrak{g} \to \mathfrak{g}l(V)$ be the associated Lie algebra homomorphism.

Claim 3.7. *Let $\mathfrak{a} \subset \rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r})$ be an ideal of $\rho(\mathfrak{g})$, and set S be the subalgebra of $\mathrm{End}(V)$ generated by 1 and x for $x \in \mathfrak{a}$. Suppose that $\mathfrak{b} \subset \mathfrak{a}$ is a $\rho(\mathfrak{g})$-ideal and $Tr_V(bs) = 0$ for all $b \in \mathfrak{b}$ and all $s \in S$. Then $\mathfrak{b} = 0$.*

Proof. Let $n = \dim_{\mathbb{C}}(V)$. Since $b \in S$, so is b^r for all $r > 0$. Since $Tr_V(bs) = 0$ for all $s \in S$, it follows that $Tr_V(b^n) = 0$ for all $n > 1$. Denote by $\{\lambda_1, \ldots, \lambda_n\}$ the diagonal entries of the Jordan canonical form of b. The coefficients of the characteristic polynomial of b are the elementary symmetric functions of the $\{\lambda_i\}$, whereas the trace of b^n is $\sum_i \lambda_i^n$. The symmetric polynomials of the λ_i form a polynomial algebra with polynomial basis the elementary symmetric polynomials. The Newton polynomials give a way to write each elementary symmetric polynomial as a polynomial in the $p_k = \sum_i \lambda_i^k$, so that the p_k also form a polynomial basis for the symmetric polynomials. It follow that $Tr_V(b^k) = 0$ for all k if and only if the characteristic polynomial for b is $b^n = 0$, which is equivalent to b being a nilpotent endomorphism of V. Since this holds for all $b \in \mathfrak{b}$, the ideal $\mathfrak{b}$ consists of nilpotent elements. By Lemma 3.7 the subalgebra $\mathfrak{b}$ acts trivially on the simple $\mathfrak{g}$-module V and hence $\mathfrak{b} = 0$. $\square$

We return to the proof of the theorem. Since $\mathfrak{r}$ is solvable, for some k, $\mathcal{D}_k(\mathfrak{r}) = 0$, and consequently, $\rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathcal{D}_k(\mathfrak{r})) = 0$. Suppose by contradiction that $\rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r}) \neq 0$. Set $\mathfrak{a} = \rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathcal{D}_t(\mathfrak{r}))$, where $t \geq 1$ is chosen to be the largest integer for which this image is non-zero. Since $\rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathcal{D}_{t+1}(\mathfrak{r})) = 0$, it follows that $\mathfrak{a} \subset \mathrm{End}(V)$ is a non-zero abelian subalgebra.

We set $\mathfrak{b} = [\rho(\mathfrak{g}), \mathfrak{a}] \subset \mathfrak{a}$. For any $x \in \rho(\mathfrak{g})$ and $y \in \mathfrak{a}$, we have $Tr_V([x, y]s) = Tr_V(xys - yxs) = Tr_V(xys - xsy) = 0$, where the last equality comes from the fact that $ys = sy$ since $\mathfrak{a}$ is abelian and s is contained in the subalgebra of $\mathrm{End}(V)$ generated by 1 and $\mathfrak{a}$. Applying the claim, we see that $\mathfrak{b} = [\rho(\mathfrak{g}), \mathfrak{a}] = 0$, showing that $\mathfrak{a}$ is in the center of $\rho(\mathfrak{g})$.

Since $\mathfrak{a}$ is contained in the center of $\rho(\mathfrak{g})$, we conclude that $\rho(\mathfrak{g})$ commutes with S. For $x, y \in \mathfrak{g}$ consider $Tr_V(\rho([x, y])s) = Tr_V(\rho(x)\rho(y)s - \rho(y)\rho(x)s) = Tr_V(\rho(x)\rho(y)s - \rho(x)s\rho(y)) = 0$, where the last equality comes from the fact that $s\rho(y) = \rho(y)s$. Since $\mathfrak{a} \subset \rho(\mathcal{D}_2(\mathfrak{g}))$, applying the claim with $\mathfrak{b} = \mathfrak{a}$, we conclude that $\mathfrak{a} = 0$. This contradicts the assumption that $\mathfrak{a} \neq 0$, establishing that $\rho(\mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r}) = 0$.

Since V is an arbitrary simple $\mathfrak{g}$-module, $\mathcal{D}_2(\mathfrak{g}) \cap \mathfrak{r} \subset \mathcal{N}$. $\square$

Corollary 3.8. *Let V be a finite dimensional complex vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a solvable sub Lie algebra. Then there is a basis for V in which $\mathfrak{g}$ is given by upper triangular matrices, or equivalently, $\mathfrak{g}$ preserves is a flag*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

with $\dim(V_i/V_{i-1}) = 1$ for every i.

Proof. By Corollary 4.6 all simple $\mathfrak{g}$-modules are 1-dimensional. Let $V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_k \supset 0$ be the Jordan Holder decomposition of V as a $\mathfrak{g}$-module. Then for each i we have $\dim(V_i/V_{i+1}) = 1$. Setting $V'_r = V_{k+1-r}$ we have a flag of $\mathfrak{g}$-modules

$$0 = V'_0 \subset V'_1 \subset V'_2 \subset \cdots \subset V'_k \subset V'_{k+1} = V,$$

where V'_i/V'_{i-1} has dimension 1 for all $1 \leq i \leq k+1$. Choose a basis *adapted* to this flag, in the sense that the basis $\{e_1, \ldots, e_{k+1}\}$ has the property that for each j the subset $\{e_1, \ldots, e_j\}$ is a basis for V'_j. In this basis $\mathfrak{g}$ is represented by upper triangular matrices. $\square$

Corollary 3.9. *Let V be a finite dimensional complex vector space and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a sub Lie algebra. Denote by $\mathfrak{r}$ the radical of $\mathfrak{g}$ and by $\mathcal{N}$ the nilradical of $\mathfrak{g}$. Then $\mathcal{N}$ and $\mathfrak{r}$ are orthogonal under B_V:*

$$B_V(\mathcal{N}, \mathfrak{r}) = (Tr_V)|_{\mathcal{N}\circ\mathfrak{r}} = 0.$$

Proof. By Corollary 4.8 there is a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ with $\dim(V_i/V_{i-1)} = 1$ for each i that is preserved by $\mathfrak{r}$. Since the nilradical of $\mathfrak{g}$ is equal to the nilradical of $\mathfrak{r}$, the ideal $\mathcal{N}$ acts trivially on all the simple $\mathfrak{r}$-modules and hence maps V_i to V_{i-1} for every i. Hence, for each i and for any $x \in \mathfrak{r}$ and $n \in \mathcal{N}$ the composition xn sends V_i to V_{i-1}, and consequently has zero trace. □

4 Cartan's Criterion for Solvablility

Cartan's criterion for solvability is the converse to Corollary 4.9.

Theorem 4.1. *(Cartan's Criterion for Solvability) Let V be a finite dimensional complex vector space, let $\mathfrak{g}$ be a Lie algebra and let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation. Then $\rho(\mathfrak{g})$ is a solvable subalgebra of $\mathfrak{gl}(V)$ if and only if $\rho(\mathcal{D}_2(\mathfrak{g}))$ is orthogonal to $\rho(\mathfrak{g})$ under B_V.*

Proof. Clearly, if suffices to consider the case when $\mathfrak{g} \subset \mathfrak{gl}(V)$ and ρ is the identity representation. Corollary 4.9 shows the necessity of the condition for $\mathfrak{g}$ to be solvable.

We consider the converse. Suppose that $\mathcal{D}_2(\mathfrak{g})$ is orthogonal under B_V to $\mathfrak{g}$.

Claim 4.2. *Let $A \subset B \subset \mathfrak{gl}(V)$ be linear subspaces and let $T = \{t \in \mathfrak{gl}(V) \mid [t, B] \subset A$. If $x \in T$ and $Tr_V(xt) = 0$ for all $t \in T$, then x is a nilpotent transformation of V.*

Proof. We write $x = x_{ss} + n_n$ the Jordan decomposition of x and we choose a diagonal basis $\{e_i\}_{i=1}^n$ of V for x_{ss}: $x_{ss}(e_i) = \lambda_i e_i$ for $1 \leq i \leq n$. Let $K \subset V$ be the rational subspace generated by the λ_i. To complete the claim we must show that the λ_i are all zero, or equivalently that $K = 0$.

Take the basis $\{E_{i,j}\}_{1\leq i,j\leq n}$ for $\mathfrak{gl}(V)$ where $E_{i,j}(e_r) = \delta(i, r)e_j$. Then $ad(x_{ss})(E_{i,j}) = (\lambda_i - \lambda_j)E_{i,j}$. Fix a $\mathbb{Q}$-linear form $f\colon K \to \mathbb{Q}$. Let $D \in \mathfrak{gl}(V)$ be the element defined by $D(e_i) = f(\lambda_i)e_i$. Then

$$ad(D)(E_{i,j}) = (f(\lambda_i) - f(\lambda_j)E_{i,j} = f(\lambda_i - \lambda_j)E_{i,j}.$$

There is a polynomial $p(t)$ with $\mathbb{Q}$ coefficients and with 0 constant term such that $p(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$ for all i, j. [If $\lambda_i - \lambda_j = \lambda_k - \lambda_\ell$, then since f is linear $f(\lambda_i) - f(\lambda_j) = f(\lambda_k) - f(\lambda_\ell)$. Similarly, if $\lambda_i - \lambda_j = 0$ then $f(\lambda_i - \lambda_j) = 0$.] Then $ad(D) = p(ad(x_{ss}))$. On the other hand, there is a polynomial with zero constant term q such that $q(ad(x)) = ad(x_{ss})$. It follows that $ad(D) = p(q(ad(x)))$. Since $ad(x)(B) = [x, B] \subset A$, it follows that $ad(D)(B) = [D, B] \subset A$, and hence $D \in T$. Thus, $Tr(Dx) = 0$. On the other hand $Tr(Dx) = \sum_i \lambda_i f(\lambda_i)$, so that $0 = f(Tr(Dx)) = \sum_i f(\lambda_i)^2$. Since the $f(\lambda_i)$ are rational numbers, this imples $f(\lambda_i) = 0$ for all i. But f was an arbitrary linear form on K and K is generated over $\mathbb{Q}$ by λ_i, so it follows that any linear form on K is trivial, showing that K is trivial. □

Now we apply this claim to our situation. Let $B = \mathfrak{g}$ and $A = \mathcal{D}_2(\mathfrak{g})$ in $\mathfrak{g}l(V)$ and T as in the claim for this $A \subset B$.. Consider $t \in T$ and $[x, y] \in \mathcal{D}_2(\mathfrak{g})$. We have $Tr_V(t[x, y]) = Tr_V(([t, x])y]$. But $[t, x] \in \mathcal{D}_2(\mathfrak{g})$, so by hypothesis, $Tr_V(([t, x])y) = 0$ for all $t \in T$. By linearity, it follows that $Tr_V(tu) = 0$ for any $u \in \mathcal{D}_2(\mathfrak{g})$ and all $t \in T$. On the other hand, it is clear that $\mathcal{D}_2(\mathfrak{g}) \subset T$. Applying the claim, we see that every element in $\mathcal{D}_2(\mathfrak{g})$ is a nilpotent transformation of V. It follows from Corollary 3.4 that $\mathcal{D}_2(\mathfrak{g})$ is a nilpotent Lie Algebra and hence $\mathfrak{g}$ is solvable. □

Corollary 4.3. *Let V be a finite dimensional vector space and $\mathfrak{g} \subset \mathfrak{g}l(V)$. Then the space of all $x \in \mathfrak{g}$ that are orthogonal under B_V to $\mathfrak{g}$ is a solvable ideal of $\mathfrak{g}$.*

Proof. Consider L the vector space of $x \in \mathfrak{g}$ orthogonal to $\mathfrak{g}$ under B_V. Since $B_V([x, y], z) = B_V(x, [y, z])$, it follows that L is an ideal in $\mathfrak{g}$. Since $B_V(L, L) = 0$ and *a fortiori* $B_V(L, \mathcal{D}_2(L)) = 0$, it follows from the previous theorem that L is solvable. □