Lie Groups: Fall, 2022
Lecture VI
Structure of Simple Lie Algebras

November 28, 2022

1 The Four Infinite Families

As we pointed out in the last lecture there are four infinite families of simple Lie algebras and apart from them there are only 5 others. In this lecture we shall discuss in more detail the infinite families, their roots, their split real forms and their compact forms.

1.1 The A-series $A_n = \mathfrak{sl}(n+1,\mathbb{C})$ for $n \geq 1$

One denotes by $A_n, n \geq 1$, the Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$. This is a Lie alebra whose Cartan subalgebra, $\mathfrak{h}$, is the space of those matrices of trace 0, all of whose off-diagonal terms are zero . The Cartan subalgebra has dimension n. For each $i \leq n+1$ let $z_i\colon \mathfrak{h} \to \mathbb{C}$ send a matrix $H \in \mathfrak{h}$ to its (i,i) entry The roots of this algebra are $\{\alpha_{i,j} = (z_i - z_j)\colon \mathfrak{h} \to \mathbb{C}\}$ where $1 \leq i, j \leq n+1$ and $i \neq j$. For any such (i,j) let $E_{i,j} \in \mathfrak{sl}(n+1,\mathbb{C})$ be the matrix with 1 in the i,j place and zero elsewhere and let $L_{i,j}$ be the one-dimensional complex subspace spanned by $E_{i,j}$. It is the root space for $\alpha_{i,j}$, and is denoted $\mathfrak{sl}(n+1,\mathbb{C})_{\alpha_{i,j}}$. Here are the basic properties:

- There is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{sl}(n+1,\mathbb{C})$.
- The non-zero eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}(n+1,\mathbb{C})$, i.e., the root spaces, are one-dimensional
- If α is a non-zero eigenvalue for the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}(n+1,\mathbb{C})$; i.e., if α is a root, then $-\alpha$ is a root.

- For each root α the subspace

$$\mathfrak{sl}(n+1,\mathbb{C})_\alpha \oplus \mathfrak{sl}(n+1,\mathbb{C})_{-\alpha} \oplus [\mathfrak{sl}(n+1,\mathbb{C})_\alpha, \mathfrak{sl}(n+1,\mathbb{C}_{-\alpha}]$$

is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

- The roots span $\mathfrak{h}^*$.

The split real form of $\mathfrak{sl}(n+1,\mathbb{C})$ is $\mathfrak{sl}(n+1,\mathbb{R})$. Its Cartan is denoted $\mathfrak{h}_\mathbb{R} = \mathfrak{h} \cap \mathfrak{sl}(n+1,\mathbb{R})$. The roots are real on $\mathfrak{h}_\mathbb{R}$.

The compact form is $\mathfrak{su}(n+1)$, which is the real subspace for the complex anti-inear involution $A \mapsto -\overline{A}^{tr}$. The roots are purely imaginary on $\mathfrak{h}_c = i\mathfrak{h}_\mathbb{R}$ the Cartan for the compact form.

1.2 The $D_n = \mathfrak{so}(2n), n \geq 4$ Series

In general $\mathfrak{so}(n)$ consists of complex orthogonal $n \times n$ matrices; i.e., those that satisfy $A + A^{tr} = 0$.

The low dimensional cases of $\mathfrak{so}(2n)$ are special. First, $\mathfrak{so}(2)$ is the one-dimensional abelian Lie algebra of matrices

$$\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$$

The Lie algebra $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and is semi-simple but not simple. It turns out that $\mathfrak{so}(6)$ is isomorphic to $\mathfrak{sl}(4)$ and already occurs as A_3 in the A-series.

Nevertheless to understand the structure of $\mathfrak{so}(2n)$ we begin with $\mathfrak{so}(4)$. The Lie algebra $\mathfrak{so}(4)$ is of rank two. A Cartan subalgebra $\mathfrak{h}$ for it consists of matrices

$$\mathfrak{h} = \begin{pmatrix} 0 & -z_1 & 0 & 0 \\ z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 \\ 0 & 0 & z_2 & 0 \end{pmatrix}.$$

There are four root spaces and they span the space the matrices in block 2×2 form

$$\begin{pmatrix} 0 & A \\ -A^{tr} & 0 \end{pmatrix}.$$

We write the vector space of the A in upper 2×2 block as $C_1 \otimes C_2^{tr}$ where the C_i are column matrices of size 2. The left action of $\mathfrak{h}$ on the upper left

2×2 block is the rotational action of upper $\mathfrak{so}(2)$ on C_1 tensor the identity on C_2^{tr}. The rotation action is

$$\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -zb \\ za \end{pmatrix}.$$

Thus, the eigenvalues for the action of $\mathfrak{h}$ on A are $\pm iz_1$, each of multiplicity two. Minus the right action of $\mathfrak{h}$ is the tensor product of the identity on C_1 with the negative of the rotational action of the lower $\mathfrak{so}(2)$ on C_2^{tr}. Its eigenvalues are $\pm iz_2$. Thus, the four roots of $\mathfrak{h}$ are $\pm iz_1 \pm iz_2$.

More generally, if we consider $\mathfrak{so}(2n)$ for $n \geq 3$, we take as the Cartan $\mathfrak{h}$ the block 2×2 diagonal matrices the j^{th} diagonal block being

$$\begin{pmatrix} 0 & -z_j \\ z_j & 0 \end{pmatrix}.$$

The root spaces are then grouped into four dimensional vector spaces indexed y $1 \leq j < k \leq n$: the 2×2 blocks in position

$$\begin{pmatrix} a_{2j-1,2k-1} & a_{2j-1,2k} \\ a_{2j,2k-1} & a_{2j,2k} \end{pmatrix}$$

The roots associated with this block are $\pm iz_j \pm iz_k$).

Thus, there are $2n(n-1)$ roots $\pm iz_j \pm iz_j$ for $1 \leq j < k \leq n$. The Cartan has dimension n giving a dimension of $n(2n-1)$ for $\mathfrak{so}(2n)$.

As before we have:

- There is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{so}(2n)$.
- The non-zero eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{so}(2n)$, i.e., the root spaces, are one-dimensional
- If α is a non-zero eigenvalue for the adjoint action of $\mathfrak{h}$ on $\mathfrak{so}(2n)$; i.e., if α is a root, then $-\alpha$ is a root.
- For each root α the subspace

  $$\mathfrak{so}(2n)_\alpha \oplus \mathfrak{so}(2n)_{-\alpha} \oplus [\mathfrak{so}(2n)_\alpha, \mathfrak{so}(2n)_{-\alpha}]$$

  is a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.
- The roots span $\mathfrak{h}^*$.

In this presentation the real subalgebra $\mathfrak{so}(2n,\mathbb{R})$ is the Lie algebra of the compact groups $Spin(2n)$ or $SO(2n)$. These are compact groups with finite fundamental groups Thus, this real form is the compact form. As we have already seen, the roots are purely imaginary on the Cartan $\mathfrak{h}_{\mathbb{R}}$ of this real form.

The split real from is $\mathfrak{so}(n,n)$, the Lie algebra of the orthogonal group of the real quadratic form

$$\sum_{i=1}^{2n}(-1)^{i+1}x_i^2=\sum_{j=1}^{n}u_jv_j.$$

In the latter form, the Cartan for this real form is the diagonal matrix with diagonal entries $\{t_1,-t_1,t_2,-t_2,\cdots,t_n,-t_n\}$ for the $t_i\in\mathbb{R}$

Notice that there are many other real forms of this complex Lie algebra, namely the $\mathfrak{so}(p,q)$ with $p+q=2n$.

1.2.1 The low dimensional special cases

In the special case when $n=2$ there are four roots $i(\pm z_1\pm z_2)$. I have left it as a homework exercise to show this algebra is isomorphic to $A_1\oplus A_1$; i.e., $\mathfrak{so}(3)\oplus\mathfrak{so}(3)$.

In the special case of $\mathfrak{so}(6)$, the roots are $i\{\pm z_1\pm z_2,\pm z_1\pm z_3,\pm z_2\pm z_3\}$. It is a homework problem to idenitfy this with $A_3=\mathfrak{sl}(4)$.

1.3 The B-series $B_n=\mathfrak{so}(2n+1); n\geq 2$

The rank of $\mathfrak{so}(2n+1)$ is n. Thus, the natural inclusion $\mathfrak{so}(2n)\to\mathfrak{so}(2n+1)$ sends the Cartan subalgebra of $\mathfrak{so}(2n)$ to one for $\mathfrak{so}(2n+1)$. Thus, all the roots of $\mathfrak{so}(2n)$, $\pm iz_j\pm iz_k$ for $1\leq j<k\leq n$, are roots of the Cartan for $\mathfrak{so}(2n+1)$, and the root spaces are the image under the embedding of the corresponding root spaces in $\mathfrak{so}(2n)$. But in addition there are the skew-symmetric matrices with zero except on the last row and last column. The left action of $\mathfrak{h}$ rotates the two dimensional subspaces in position $(2j-1,2n+1),(2j,2n+1)$ with eigenvalues $\pm iz_j$ and leaves the last row unchanged. Minus the right action of $\mathfrak{h}$ leaves the last column unchanged and rotates the plane in positions $(2n+1,2i-1),2n+1,2i)$ with eigenvalues $\mp iz_j$. Thus, the roots associated with these spaces are $\pm iz_j; 1\leq j\leq n$. This gives us a total of $2n(n-1)+2n=2n^2$ roots and a Cartan of rank n for a total dimension of $(2n+1)n$ for $\mathfrak{so}(2n+1)$.

As before we have:

- There is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{so}(2n+1)$.
- The non-zero eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{so}(2n+1)$, i.e., the root spaces, are one-dimensional
- If α is a non-zero eigenvalue for the adjoint action of $\mathfrak{h}$ on $\mathfrak{so}(2n+1)$; i.e., if α is a root, then $-\alpha$ is a root.
- For each root α the subspace

$$\mathfrak{so}(2n+1)_\alpha \oplus \mathfrak{so}(2n+1)_{-\alpha} \oplus [\mathfrak{so}(2n+1)_\alpha, \mathfrak{so}(2n+1)_{-\alpha}]$$

  is a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.
- The roots span $\mathfrak{h}^*$.

The real form in the given presentation is $\mathfrak{so}(2n+1,\mathbb{R})$ which is the compact real from. The split real form is $\mathfrak{so}(n+1,n)$ which contains $\mathfrak{so}(n,n)$ as a subgroup of the same rank.

1.4 The C-series $C_n = Sp(2n)$ for $n \geq 2$

Consider the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. This is the standard symplectic form in dimension $2n$ and the complex symplectic group is defined as $A \in GL(2n,\mathbb{C})$ that preserve this bilinear form; i.e., with the property that $A^{tr}JA = J$. Differentiating at the identity we see that the complex symplectic Lie algebra $\mathfrak{sp}(2n)$ is

$$\{X \in \mathbb{C}[2n] \,|\, X^{tr}J + JX = 0\}.$$

Direct computation shows that this is the group of matrices written in $n \times n$ block form that satisfy

$$\begin{pmatrix} A & B \\ C & -A^{tr} \end{pmatrix}$$

where B and C are symmetric, i.e., $C^{tr} = C$ and $B^{tr} = B$.

We take as the Cartan subalgebra the matrices in $\mathfrak{sp}(2n)$ the matrices where A is diagonal and B and C are zero. Let $\lambda_1, \ldots, \lambda_n$ record the diagonal entries of A.

Then the non-zero eigenspaces for this Cartan are:

- For $1 \leq i \neq j \leq n$, the element $X_{i,j} = E_{i,j} - E_{n+j,n+i}$ is an eigenvector for $\mathfrak{h}$ with eigenvalue $\lambda_i - \lambda_j$

- For $1 \leq i < j \leq n$, the element $Y_{i,j} = E_{i,n+j} + E_{j,n+i}$ is an eigenvector with eigenvalue $\lambda_i + \lambda_j$

- for $1 \leq i < j \leq n$, the element $Z_{i,j} = E_{n+i,j} + E_{n+j,i}$ is an eigenvector with eigenvalue $-\lambda_i - \lambda_j$

- For $1 \leq i \leq n$, the element $U_i = E_{i,n+i}$ has eigenvalue $2\lambda_i$

- For $1 \leq i \leq n$, the element $V_i = E_{n+i,i}$ has eigenvalue $-2\lambda_i$.

Thus, the roots for $\mathfrak{s}p(2n)$ are

$$\{\pm(\lambda_i - \lambda_j), \pm(\lambda_i + \lambda_j) \ \text{ for } \ 1 \leq i < j \leq n \ \text{ and } \ \pm 2\lambda_i \ \text{ for } \ 1 \leq i \leq n\}.$$

There are $2n^2$ roots and the Cartan of rank n for a total dimension of $n(2n+1)$.

As before, we have:

- There is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{s}p(2n)$.

- The non-zero eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{s}p(2n)$, i.e., the root spaces, are one-dimensional

- If α is a non-zero eigenvalue for the adjoint action of $\mathfrak{h}$ on $\mathfrak{s}p(2n)$; i.e., if α is a root, then $-\alpha$ is a root.

- For each root α the subspace

$$\mathfrak{s}p(2n)_\alpha \oplus \mathfrak{s}p(2n)_{-\alpha} \oplus [\mathfrak{s}p(2n)_\alpha, \mathfrak{s}(2n)_{-\alpha}]$$

  is a subalgebra isomorphic to $\mathfrak{s}l(2, \mathbb{C})$.

- The roots span $\mathfrak{h}^*$.

In this presentation, the real symplectic Lie algebra is the split real from (since the eligenvalues are real on the Cartan). It is the Lie algebra of the usual symplectic group $SP(2n)$ of automorphisms of $\mathbb{R}^{2n}$ preserving the symplectic form (i.e., skew symmetric bilinear form) given by J. The compact form of the symplectic group is $SP(2n, \mathbb{C}) \cap SU(2n)$. Its Lie algebra is

$$\{A \in \mathfrak{s}p(2n) \,|\, -\overline{A}^{tr} = A\}.$$

One sees directly that this complex anti-linear involution preserves $\mathfrak{s}p(2n)$ and hence defines a real form of $\mathfrak{s}p(2n)$. Since the intersection of this real from with $\mathfrak{h}$ is $i\mathfrak{h}_{\mathbb{R}}$ (where $\mathfrak{h}_{\mathbb{R}}$ is the Cartan of the split real form), it follows that this real form is the compact real form. [Also, it is clear that $Sp(2n) \cap SU(2n)$ is a closed subgroup of $SU(2n)$ and hence is compact.]

2 The General Structure of Root Systems of Simple Lie Algebras

A central tool for understanding semi-simple Lie Algebras is the Killing Form.

Definition 2.1. Let L be a complex Lie Algebra. The Killing form $B\colon L \otimes L \to \mathbb{C}$ is the symmetrtic bilinear form defined by

$$B(X, Y) = \mathrm{Trace}(ad(X) \circ ad(Y))$$

for $X, Y \in L$.

The following fundamental result is a central one in the theory of semi-simple Lie algbras.

Theorem 2.2. *The Killing form of a (finite dimensional, complex) Lie algebra L is non-degenerate if and only if L is semi-simple. The Killing form is a real bilinear pairing on any real form of L. Assuming that L is semi-simple, the Killing form is positive definite on the split real form of L and is purely imaginary on the compact real form.*

We have seen that this result holds for $\mathfrak{s}l(n, \mathbb{C})$. It is a similar, straight-forward computation to prove it for each of the series A_n, B_n, C_n, D_n as defined above. But the proof in general takes us pretty far afield. I will establish it in an appendix. For now we assume it.

We also need a result that we have discussed before.

Lemma 2.3. *The adjoint representation preerves B in the sense that*

$$B([X, Y], Z]) + B(Y, [X, Z]) = 0.$$

Definition 2.4. Let $\mathfrak{g}$ be a semi-simple Lie algebra. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan* subalgebra if it is an abelian sub-algebra consisting of elements which map to semi-simple elements (i.e., diagonalizable elements) under the adjoint representation and if $\mathfrak{h}$ is maximal with respect to these two properties.

Theorem 2.5. *Every semi-simple algebra has a non-zero Cartan sub-algebra.*

I will postpone the proof of this result.

For the rest of this section $\mathfrak{g}$ is a semi-simple complex Lie algebra. We fix a Cartan subalgebfra $\mathfrak{h} \subset \mathfrak{g}$ giving a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where Φ is the *set of roots* and $\mathfrak{g}_\alpha$ is the root space associated to the root α.

Theorem 2.6. *In the above decoposition every α is a non-zero homomorphism of $\mathfrak{h} \to \mathbb{C}$. That is to say the subalgebra of elements commuting with every element of $\mathfrak{h}$ is $\mathfrak{h}$ itself. Said another way* 0 *is not a root of* $\mathfrak{g}$.

This is a third result whose proof we postpone.

Lemma 2.7. *The roots span $\mathfrak{h}^*$ over $\mathbb{C}$.*

Proof. If not, then the roots span a proper subspace of $\mathfrak{h}^*$ and hence there is a non-zero element $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all roots α. This implies that h commutes with every root space $\mathfrak{g}_\alpha$. Of course, h also commutes with $\mathfrak{h}$ and hence h is contained in the center of $\mathfrak{g}$. This means that the subspace spanned by h is a one-dimensional, abelian ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple it is dimension gt least 2, so that the span of h is a non-trivial ideal. This contradicts the fact that the only non-trivial ideals of $\mathfrak{g}$ themselves semi-simple and hence not of dimension 1. □

2.1 The Killing Form and Pairs of Roots

We fix a semi-simple Lie algebra $\mathfrak{g}$ with Cartan $\mathfrak{h}$ and roots α.

Lemma 2.8. *If α and β are eigenvalues of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ and $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under the Killing form B.*

Proof. Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then $ad(X) \circ ad(Y)$ sends the γ-eigenspace for $\mathfrak{h}$ to its $\gamma+\alpha+\beta$ eigenspace. Since $\alpha+\beta \neq 0$, this composition has trace zero. □

Corollary 2.9. *Under the Killing form $\mathfrak{h}$ is orthogonal to all the root spaces. An eigenspace space $\mathfrak{g}_\alpha$ is B-orthogonal to all eigenspaces except the $-\alpha$ eigenspace. As a result the restriction of the Killing form to $\mathfrak{h}$ is non-degenerate. Also for each root α the eigenspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are dually paired under the Killing form.*

2.2 The Killing Form and the Cartan

We can use the Killing form to define a non-degenerate pairing on $\mathfrak{h}^*$. Given $a, b \in \mathfrak{h}^*$ we write $a = (h_a, \cdot)$ for a unique $h_a \in \mathfrak{h}$. Then the dual pairing $B^*(a, b) = b(h_a)$. Clearly, B^* is a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$. We denote the dual pairing on $\mathfrak{h}^*$ by $\langle a, b\rangle$. In particular for a root α we have $\langle \alpha, \alpha\rangle = \alpha(t_\alpha) = B(t_\alpha, t_\alpha)$.

Definition 2.10. For each root α let $t_\alpha \in \mathfrak{h}$ be the unique element such that for each $h \in \mathfrak{h}$ we have

$$B(t_\alpha, h) = \alpha(h).$$

Notice that since B and α are both real on split real form it follows that t_α is contained in the Cartan $\mathfrak{h}_{\mathbb{R}}$ of the split real form.

Lemma 2.11. *For $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$, we have*

$$[X, Y] = B(X, Y)t_\alpha.$$

Proof. For $h \in \mathfrak{h}$, we have

$$B(h, [X, Y]) = B([h, X], Y) = B(\alpha(h)X, Y) = \alpha(h)B(X, Y) = \langle h, t_\alpha\rangle B(X, Y) = \langle h, B(X, Y)t_\alpha\rangle.$$

Since this is true for all $h \in \mathfrak{h}$ and since $B|_{\mathfrak{h}}$ is non-degenerate, this implies

$$[X, Y] = B(X, Y)t_\alpha.$$

□

Corollary 2.12. *We have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}(t_\alpha) \subset \mathfrak{h}$.*

Proof. The inclusion $\subset$ is clear from the previous lemma. That the image is all of $\mathbb{C}(t_\alpha)$ folllows from the non-degneracy of the pairing. □

2.3 The $\mathfrak{s}l(2)$ associated with a root

Consider a root α and a non-zero element $X_0 \in \mathfrak{g}_\alpha$. Then there is $Y_0 \in \mathfrak{g}_{-\alpha}$ with $B(X_0, Y_0) = t_\alpha$. Then for the 3-dimensional Lie algebra spanned (over $\mathbb{C}$) by X_0, Y_0, t_α

Proposition 2.13. $\alpha(t_\alpha) = B(t_\alpha, \alpha) \neq 0$

Proof. Suppose that $\alpha(t_\alpha) = 0$. Then the 3-dimensional Lie algebra with $\mathbb{C}$-basis X_0, Y_0, t_α has the following Lie bracket structure:

- $[X_0, Y_0] = t_\alpha$
- $[t_\alpha, X] = \alpha(t_\alpha)X_0 = 0$
- $[t_\alpha, Y_0] = -\alpha(t_\alpha)Y_0 = 0$
- $[X_0, Y_0] = t_\alpha$.

This is a solvable Lie algebra with t_α in the commutator subgroup.

Proposition 2.14. *Let V be a finite dimensional complex vector space and let $L \subset \mathfrak{g}l(V)$ be a solvable Lie subalgebra. Then there is a flag*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_K = V,$$

with the property that for each $i \geq 1$ the quotient V_i/V_{i-1} has dimension 1, a flag that is preserved by the L action. Furthermore, for each $i \geq 1$ the derived subalgebra $\mathcal{D}_2(L) = [L, L]$ sends V_i to V_{i-1}. In particular every element of $\mathcal{D}_2(L)$ is a nilpotent endomorphism of V.

Again, I postpose the proof of this proposition

Since t_α is in the commutator subgroup of the solvable Lie subalgebra of $\mathfrak{g}$ with vector space basis $\{X, Y, t_\alpha\}$, the previous proposition implies that the action of t_α on $\mathfrak{g}$ is nilpotent. This contradicts the fact that since $t_\alpha \in \mathfrak{h}$ it is a non-zero semi-simple element. (And $t_\alpha \neq 0$ since it is dual under the Killing form to a root $\alpha \neq 0$.) This contradiction proves that $\alpha(t_\alpha) = B(t_\alpha, t_\alpha) \neq 0$. □

Now we set $h_\alpha = 2t_\alpha/B(t_\alpha, t_\alpha)$ and $Y = 2Y_0/B(t_\alpha, t_\alpha)$ and $X = X_0$. Then the 3-dimensional algebra with generators X, Y, h_α has the following Lie bracket relations:

- $[h_\alpha, X] = 2X$
- $[h_\alpha, Y] = -2Y$,
- $[X, Y] = h_\alpha$.

That is to say, this subalgebra is isomorphic to $\mathfrak{s}l(2, \mathbb{C})$. We denote this subalgebra by $\mathfrak{s}l(2)_\alpha$. *Apriori.* this subalgebra may depend upon the choice of $X \in \mathfrak{g}_\alpha$, but we shall see that it does not.

Now consider the subalgebra $\mathfrak{m} = \mathfrak{h} \oplus_{c \in \mathbb{C}^*} \mathfrak{g}_{c\alpha}$. It is a module over $\mathfrak{s}l(2)_\alpha$. By the structure of these modules, we see that $\mathfrak{m}$ is a direct sum of towers. The towers with even eigenvalues for h_α each containing a one-dimensional subspace on which h_α acts trivially and different such towers lead to linearly independent elements in the 0 eigenspace of h_α. In $\mathfrak{m}$, the zero eigenspace of t_α is $\mathfrak{h}$. On the other hand, $\mathfrak{s}l(2,\mathbb{C})_\alpha + \mathfrak{h}$ is a $\mathfrak{s}l(2,\mathbb{C})_\alpha$ submodule. This shows that the towers with even $\mathfrak{h}_\alpha$ eigenvalues consist of the direct sum $\mathfrak{s}l(2,\mathbb{C})_\alpha$. From this it follows that $\mathfrak{g}_\alpha$ is one dimensional and that 2α is not a root.

Now let us consider the towers with odd eigenvalues for h_α. If there is such a tower, then $\alpha/2$ must be a root. But we have already seen that twice a root is never a root, so since α is a root, $\alpha/2$ cannot be a root. Consequently, $\mathfrak{m} = \mathfrak{s}l(2,\mathbb{C})_\alpha \oplus \mathrm{Ker}(\alpha)$. From this it follows immediately that $\mathfrak{s}l(2,\mathbb{C})_\alpha$ is independent of the choice of $X \in \mathfrak{g}$.

Corollary 2.15. *If α is a root, then $c\alpha$ is a root for $c \in \mathbb{C}^*$ if and only if $c = \pm 1$. The root spaces are one-dimensional.*

2.4 The Involution

We can use the $\mathfrak{s}l(2,\mathbb{C})_\alpha$ in another important way. The action of $\mathfrak{s}l(2,\mathbb{C})_\alpha$ acts on $\mathfrak{g}$. This leads to an action of $SL(2,\mathbb{C})_\alpha$ on $\mathfrak{g}$. In $SL(2,\mathbb{C})_\alpha$ the matrix

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

sends h_α to $-h_\alpha$ and is the identity on $\mathrm{Ker}(\alpha) \subset \mathfrak{h}$. Notice that this involution also interchangeas $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Notice, that being an automorphism of the Lie algebra $\mathfrak{g}$ stabilizing $\mathfrak{h}$ it also stabilizes the set of roots. The formula for this involution of $\mathfrak{h}$ is

$$h \mapsto h - \frac{2\langle \alpha, h\rangle}{\langle \alpha, \alpha\rangle}.$$

2.5 The rational form of $\mathfrak{h}^*$

Lemma 2.16. *If α and β are roots then*

$$\frac{2\beta(h_\alpha)}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}.$$

Proof. We need only consider the case when $\beta \neq \pm\alpha$. Let

$$\mathfrak{m}_\beta = \oplus_{i\in\mathbb{Z}} \mathfrak{g}_{\beta+i\alpha}.$$

By what Corollary 2.15, it follows that $\beta + i\alpha \neq 0$ for all i, so that all the non-zero spaces are in this direct sum decomposition are one dimensional.

Furthermore, $(\beta + i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i$. Since the eigenspaces of h_α are all one dimensional and have eigenvalues differing by 2, it follows from the classification of $\mathfrak{sl}(2, \mathbb{C})$ modules that $\mathfrak{m}_\beta$ is an irreducible $\mathfrak{sl}(2\mathbb{C})_\alpha$ module. Thus, there are integers $r \leq q$ such that the eigenvalues for $\mathfrak{m}_\beta$ of h_α

$$\beta(h_\alpha) + 2q, \beta(h_\alpha) + 2q - 2, \ldots \beta(h_\alpha) + 2r,$$

and $\beta(h_\alpha) + 2r = -\beta(h_\alpha) - 2q$, from which we deduce that $\beta(h_\alpha) = q - r$ is an integer. Since

$$h_\alpha = 2t_\alpha / B(t_\alpha, t_\alpha) = 2t_\alpha / \langle \alpha, \alpha \rangle$$

we conclude that

$$\beta(h_\alpha) = \frac{2\beta(t_\alpha)}{\langle \alpha, \alpha \rangle} = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

□

Since the roots span $\mathfrak{h}^*$, there is a subset $\{\alpha_1, \ldots, \alpha_k\}$ which are a $\mathbb{C}$-basis for $\mathfrak{h}^*$

Proposition 2.17. *Fix a set of roots $\{\alpha_1, \ldots, \alpha_k\}$ that are a $\mathbb{C}$-basis for $\mathfrak{h}^*$. Set $\mathfrak{h}^*_{\mathbb{Q}}$ equal to the $\mathbb{Q}$-span of this set of roots. Then:*

- *All roots are contained in $\mathfrak{h}^*_{\mathbb{Q}}$.*
- *The subspace of $\mathfrak{h}$ that is the $\mathbb{Q}$-dual to $\mathfrak{h}^*_{\mathbb{Q}}$ is the $\mathbb{Q}$-span of the $\{h_{\alpha_i}\}$. It includes h_α for every root α.*
- *The restriction of the Killing form to $\mathfrak{h}_{\mathbb{Q}}$ is rational and positive definite. Dually, the restriction of the form dual to the Killing form to $\mathfrak{h}^*_{\mathbb{Q}}$ is also rational and positive definite*

Proof. For any root β, we have $\langle \beta, \beta \rangle = B(t_\beta, t_\beta) = Tr(ad(t_\beta)^2)$. But $ad(t_\beta)$ preserves each root space and acts on it by multiplication by $\alpha(t_\beta) = \langle \alpha, \beta \rangle$. Thus, $Tr(ad(t_\beta)^2) = \sum_\alpha \langle \alpha, \beta \rangle^2$ and so $\langle \beta, \beta \rangle = \sum_{\alpha \in \Phi} \langle \alpha, \beta \rangle^2$.This gives

$$\frac{1}{\langle \beta, \beta \rangle} = \sum_\alpha \frac{\langle \alpha, \beta \rangle^2}{\langle \beta, \beta \rangle^2}.$$

The right-hand side is a sum of squares of half-integers and hence is a non-negative rational number. On the other hand, we know that $\langle \beta, \beta \rangle \neq 0$. We conclude that $\langle \beta, \beta \rangle$ is a positive rational number. This is true for every root.

For roots α, β we have $2\langle \beta, \alpha \rangle = n_{\beta,\alpha} \langle \alpha, \alpha, \rangle$ where $n_{\beta,\alpha}$ is the integer

$$n_{\beta,\alpha} = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

It follows that $\langle \beta, \alpha \rangle \in \mathbb{Q}$ for all roots α, β.

Lastly we must show that every root β is in the $\mathbb{Q}$-span of $\{\alpha_1, \ldots, \alpha_k\}$. Since the α_i are a $\mathbb{C}$-basis of $\mathfrak{h}^*$ we have $\beta = \sum_i c^i \alpha_i$ with the c^i complex numbers. Thus,

$$\frac{\langle \beta, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \sum_i \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} c^i.$$

Consider

$$A_{i,j} = \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

It is a matrix with rational coefficients, and since the α_i form a $\mathbb{C}$-basis, this matrix is invertible over $\mathbb{C}$. Thus, it is invertible over $\mathbb{Q}$. Thus, we have

$$n_{\beta,\alpha_j} (A_{i,j})^{-1} = c^i.$$

Since the n_{β,α_j} are integers and $A_{i,j}^{-1}$ is a rational matrix, it follows that the $c^i \in \mathbb{Q}$. This proves that all roots are in $\mathfrak{h}^*_{\mathbb{Q}}$.

Let $\mathfrak{h}_{\mathbb{Q}}$ be the $\mathbb{Q}$ subspace of $\mathfrak{h}$ that is the $\mathbb{Q}$-dual to $\mathfrak{h}^*_{\mathbb{Q}}$. Then for any $h \in \mathfrak{h}_{\mathbb{Q}}$, non-zero, we have

$$B(h, h) = \sum_\alpha \alpha(h)^2.$$

This is a sum of squares, which because $h \in \mathfrak{h}_{\mathbb{Q}}$ are all rational squares, and since the roots span $\mathfrak{h}^*$, at least one of the rational numbers is non-zero. It follows that $B(h, h) > 0$. This proves that B is positive definition on $\mathfrak{h}_{\mathbb{Q}}$.

We have already seen that for all pairs of roots α and β, we have $\alpha(h_\beta) \in \mathbb{Z}$. This proves that for all roots β, the element $h_\beta \in \mathfrak{h}_{\mathbb{Q}}$. $\square$

2.6 Properties of Root Systems for semi-Simple Lie Algebras

Let $\mathfrak{g}$ be a semi-simple complex Lie algebras. Here are the properties about the Cartan subalgebra and the Killing form

- There is a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$.
- The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is semi-simple and decomposes $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$,

- The Killing form B on $\mathfrak{g}$ is non-degenerate as is its restriction to $\mathfrak{h}$.
- The restriction of the Killing form to $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is non-degenerate and dually pairs $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$. where Φ is the set of roots.
- Each root space $\mathfrak{g}_\alpha$ is one-dimensional.
- For each root α there is an involution I_α of $\mathfrak{g}$ that stabilizes $\mathfrak{h}$, sends h_α to $-h_\alpha$ and is the identity on $\mathrm{Ker}(\alpha) \subset \mathfrak{h}$.
- the restriction of the Killing form to the rational subspace spanned by the $\{h_\alpha\}$ is rational and positive definite.

Here are the basic properties about the roots:

- The set of roots Φ spans $\mathfrak{h}^*_{\mathbb{Q}}$ and each root is non-zero.
- If α is a root then for $c \in \mathbb{C}^*$ $c\alpha$ is a root if and only if $c = \pm 1$.
- For each root α the dual pairing $I^*_\alpha \colon \mathfrak{h}^*_{\mathbb{Q}} \to \mathfrak{h}^*_{\mathbb{Q}}$ maps roots to roots and for each root β we have
$$I^*_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$
- If α and β are roots then
$$\frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}$$
is an integer.

Definition 2.18. Let V be a finite dimensional rational vector space with a positive definite rational inner product B and let $\Phi \subset V^*$ be a finite set spanning V^*. Suppose that Φ satisfies the above four properties (where $\mathfrak{h}_{\mathbb{Q}}$ with the Killing form is replaced by (V, B)). Then (V, B, Φ) is called a *root system*. In the case when $V = \mathfrak{h}_{\mathbb{Q}}$ of a semi-simple Lie algebra with its killing form the *root system of the semi-simple algebra.*

One can show that every root system decomposes as an orthogonal direct sum of indecomposable root systems. The latter are called *simple* root systems. Tho simple root systems are classified. They fall into four infinite families which are the root systems of the Lie algebras in the A_n, B_n, C_n, and D_n series and 5 sporadic root systems labeled G_2, F_4, E_6, E_7, and E_8. These are the root systems of sporadic Lie algebras (and hence simply connected, sporadic Lie groups) labeled the same way.