Lie Groups: Fall, 2022
Lecture IIIB: Lie Groups from Lie Algebras

October 7, 2022

1 Constructing a neighborhood of the identity in a Lie Group from its Lie Algebra

The work of the last two lectures leads to a construction of a local Lie sub group of a Lie group from its Lie algebra.

Theorem 1.1. *Let G be a real Lie group. Let $L = \mathfrak{g}$, and choose U in Theorem 1.14 of Lecture IIIA such that* $\exp|_U$ *is a diffeomorphism onto an open subset of G. Then* $\exp\colon U \mapsto \exp(U)$ *is an embedding of the local Lie group of $\mathfrak{g}$ onto a local Lie subgroup of G that is a neighborhood of the identity in G. The inverse isomorphism of local Lie groups is* log.

Proof. Take $L = \mathfrak{g}$ in Theorem 1.14 in Lecture IIIA and choose U as in the statement and also sufficiently small so that $\exp|_U$ is a diffeomorphism onto an open subset of G. Clearly $\exp(0) = e$, $\exp(-A) = \exp(A)^{-1}$ and

$$\exp(H(A, B)) = \exp(\log(\exp(A)\exp(B))) = \exp(A)\exp(B).$$

This shows (assuming the absolute convergence of the power series for H) the exponential mapping exp embeds the local Lie algebra $\mathfrak{g}$ onto a local Lie subgroup of G that is a neighborhood of the identity. The inverse isomorphism of local Lie groups is log. □

Corollary 1.2. *Any finite dimensional real Lie group is a Lie group in the real analytic category in the sense that the underlying manifold has a real analytic structure in which the inverse and multiplication are real analytic maps.*

Proof. The real analytic structure is given by open sets of the form gU where U is as in the previous corollary and the mapping of $gU \to G$ is the left translation by g of the restriction to U of the exponential mapping. The local Lie group properties imply that on the overlap these coordinate patches define the same real analytic structure. It is clear that with respect to these coordinate patches left multiplication and inverse are real analytic maps. $\square$

2 Faithful, Finite Dimensional Linear Representation of L

A corollary of the PBW Theorem is that a Lie algebra has a faithful representation as automorphisms of an (infinite dimensional) vector space, namely its universal enveloping algebra. Even for finite dimensional L this representation is often (usually) infinite dimensional. Ado's theorem says that every finite dimensional Lie algebra over a field of characteristic zero has a faithful, finite dimensional linear representation.

Theorem 2.1. *(Ado's Theorem) Every finite dimensional Lie algebra over a field of characteristic* 0 *has a faithful finite dimensional representation.*

Proof. (Sketch) First notice that the representation $ad_L\colon L \to \mathrm{End}(L)$ is a finite dimensional representation whose kernel is the center of L, i.e., the sub Lie algebra consisting of all $X \in L$ such that $[X, Y] = 0$ for all $Y \in L$.

Thus, to complete the proof we need only construct a finite dimensional linear representation $\rho\colon L \to \mathrm{End}(V)$ whose restriction to the center of L is faithful, for then $ad_L \oplus \rho$ will be the required faithful, finite dimensional representation.

First notice that since the center is an abelian algebra, it is a finite direct sum of one-dimensional algebras. A 1-dimensional algebra has a two-dimensional representation by nilpotent matrices, namely

$$t \mapsto \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}.$$

Taking the direct sum of these gives a faithful representation of the center of L by nilpotent matrices.

To prove Ado's theorem one shows that there is a representation of L (which in fact represents the largest nilpotent ideal of L by nilpotent matrices) whose restriction to the center of L is the given faithful nilpotent representation. This is done first by induction over a solvable series to extend to the radical of L (the largest solvable ideal in L). Then there is an extension over all of L using the fact that L modulo its radical is a direct sum of simple Lie algebras, simple in the sense that have no non-trivial ideals. $\square$

3 Producing Lie Groups from Lie Algebras

We just showed that the local structure of a real Lie group can be recovered from its Lie algebra. But this leaves open the question as to whether every finite dimensional real Lie algebra is the Lie algebra of a Lie group.

Theorem 3.1. *Let L be a finite dimensional real Lie algebra. Then there is a simply connected Lie group G with $\mathfrak{g}$ isomorphic to L.*

Proof. By Ado's theorem L is a subalgebra of $\mathfrak{g}l(n,\mathbb{R})$ for some $\mathbb{R}$. The Baker-Campbell-Hausdorff formula produces an open subset $U_0 \subset \mathrm{gl}(n,\mathbb{R})$ invariant under $X \mapsto \theta(X) = -X$, together with a convergent power series $H(A,B)$ defining a map $m\colon U_0 \times U_0 \to \mathfrak{g}l(n,\mathbb{R})$ such that setting $\Omega \subset \mathfrak{g}l(n,\mathbb{R}) \times \mathfrak{g}l(n,\mathbb{R})$ the quintuple $(U, 0, \theta, \Omega, m)$, where m is the restriction of $H(A,B)$ to Ω, is a local Lie group. Choosing U_0 sufficiently small, the exponential mapping gives an isomorphism from this local Lie group to a neighborhood of the identity of $GL(n,\mathbb{R})$. Set $U_L = U_0 \cap L$. then since $H(A,B)$ is a convergent series whose terms are iterated brackets of A and B, the map m restricts to give a map $m_L\colon U_L \times U_L \to L$. We set $\Omega_L = \Omega \cap (L \times L)$. Then we have the local Lie group $(U_L, 0, \theta, \Omega_L, m_L)$.

The restriction of the exponential mapping for $\mathfrak{g}l(n,\mathbb{R})$ to U_L embeds this local Lie group as a sub local Lie group of $GL(n,\mathbb{R})$.

Next, we invoke the extension result, Theorem 3.1 of Lecture III we see that there is a connected Lie group H with $(U_L, 0, \theta, \Omega_L, m_L)$ as neighborhood of the identity and a morphism of Lie groups $H \to GL(n,\mathbb{R})$ whose differential at the identity is an isomorphism between $\mathfrak{h}$ and L. This is a Lie group with Lie algebra isomorphic to $\mathfrak{h}$.

Let $\widetilde{H} \to H$ be the universal covering. Points of $\widetilde{H}$ are equivalence classes of paths $\omega\colon [0,1] \to H$ with $\omega(0) = e$. Two paths ω and ω' are equivalent if $\omega(1) = \omega'(1)$ and if the loop $\omega * (\omega')^{-1}$ is homotopically trivial. It is left to the problems to show that $\widetilde{H}$ has a unique Lie group structure with identity element being the point represented by the constant path at the idenity of H so that the covering projection is a Lie group homomorphism.

The Lie group $\widetilde{H}$ is a simply connected Lie group with Lie algebra $\mathfrak{h}$. $\square$

Remark 3.2. The construction started with a Lie algebra $\mathfrak{h}$ and produced a sub Lie group $H \subset GL(n, \mathbb{R})$ for some n with Lie algebra isomorphic to $\mathfrak{h}$. To get a simply connected Lie group with the same Lie algebra we passed to the universal covering $\widetilde{H}$ of H. This group does not come equipped with an embedding into $GL(m, \mathbb{R})$ for any m. Indeed, there are examples of Lie algebras for which any simply connected Lie group with the given Lie algebra has no faithful, finite dimensional linear representation. One example is $SO(2,1) = PSL(2, \mathbb{R})$. The fundamental group of $PSL(2, \mathbb{R})$ is $\mathbb{Z}$ and the universal covering $\widetilde{PSL}(2, \mathbb{R})$ has no faithful finite dimensional linear representation. Of course, it has a finite dimensional linear representation with kernel the fundamental group of $\widetilde{P}SL(2, \mathbb{R})$, and even one with image $SL(2, \mathbb{R})$.

3.1 Morphisms

Next, we shall show that the construction of simply connected Lie groups from Lie algebras is functorial. As a first step in this direction we have the following:

Proposition 3.3. *Let G be a simply connected Lie group and let H be a Lie group. Given a Lie algebra map $\rho\colon \mathfrak{g} \to \mathfrak{h}$ there is a unique Lie group homomorphism $G \to H$ whose induced map on the Lie algebras is ρ.*

Proof. By the last corollary of Lecture IIIA there are neighborhoods $U_0 \subset \mathfrak{g}$ and $U_0' \subset \mathfrak{h}$ of zero with $\rho\colon U_0 \to U_0'$ is a morphism of the local Lie groups on U_0 and U_0' defined by the (convergent) BCH series.

Restricting to smaller neighborhoods U_0 and U_0' if necessary, by the Theorem 1.1 the exponential mappings embed $U_0 \to U \subset G$ and $U_0' \to U' \subset H$ and are isomorphisms of local groups from the local group structure on U_0 defined by BCH to the local group structure on U defined by G. Similarly for H. This defines a local group morphism from $\varphi\colon U \to U'$ whose differential at the identity is ρ.

Let $W \subset U$ be a neighborhood of the identity, with $W = W^{-1}$, such that $W^3 \subset U$ in the sense that for any $w_1, w_2, w_3 \in W$ all pairs

$$(w_1, w_2), (w_2, w_3), (w_1, m(w_2, w_3), (m(w_1, w_2), w_3)$$

are contained in Ω. The associative law then follows for these three elements. We define a foliation on $G \times H$. The local leaves are of the form $A(g, h) = \{(gw, h\varphi(w))\}_{w \in W}$ where the topology and differential structure on $A(g, h)$ is induced from that of G by the projection onto the first factor. (Notice that projection to the first factor is a one-to-one map on every $A(g, h)$.) Since φ is a smooth map, we see that the inclusion of $A(g, h) \subset G \times H$ is a smooth embedding onto a locally closed submanifold and the tangent planes of $A(g, h)$ vary smoothly as we vary (g, h) smoothly in $G \times H$. Thus, to show that these local leaves define a global foliation on $G \times H$ we need only see that they are compatible along their intersection. That is the content of the next claim.

Claim 3.4. *If $(g', h') \in A(g_1, h_1) \cap A(g_2, h_2)$ then this intersection contains a neighborhood N of (g', h') in both $A(g_1, h_1)$ and $A(g_2, h_2)$ and the topologies and smooth structures induced on N from $A(g_1, h_1)$ and $A(g_2, h_2)$ agree.*

Proof. We have $w_1, w_2 \in W$ with $g_1 w_1 = g' = g_2 w_2$ and $h_1 \varphi(w_1) = h' = h_2 \varphi(w_2)$. From the first pair of equations we conclude that $g_2 = g_1(w_1 w_2)$ and from the second that $h_2 = h_1 \varphi(w_1) \varphi(w_2)^{-1}$. But since $W = W^{-1}$ and $W^2 \in U$, it follows that $\varphi(w_2) = \varphi(w_2^{-1})$. Thus, we rewrite the second equation as $h_2 = h_1 \varphi(w_1) \varphi(w_2^{-1})$.

Now there is an open neighborhood Z of e such that $Zw_2 \subset W$ and $Zw_1 \in W$ for all $w \in Z$. We asert that the following equation holds:

$$(g_1(w_1 w), h_1 \varphi(w_1 w)) = (g_2(w_2 w), h_2 \varphi(w_2 w)).$$

This will establish the claim. Clearly $g_1(w_1 w) = g_2(w_2 w)$ and $h_2 \varphi(w_2 w) = h_1(\varphi(w_1)\varphi(w_2^{-1}))\varphi(w_2 w)$. Using the fact that w_1, w_2^{-1}, and $w_2 w$ are all in W and $W^3 \subset U$, we see that $(\varphi(w_1)\varphi(w_2^{-1}))\varphi(w_2 w) = \varphi(w_1 w)$ as required. □

Let $\mathcal{L}$ be the global leaf of this foliation through (e, e). We give $\mathcal{L}$ the leaf topology, where the open sets are unions of open subset of the $A(g, h)$ contained in the leaf. With this topology $\mathcal{L}$ is one-to-one smoothly immersed in $G \times H$. Notice that the projection to the first factor gives a smooth map $\mathcal{L} \to G$ that is a local diffeomorphism. Furthermore, for $g \in G$ the pre-image of gW in $\mathcal{L}$ is the disjoint union indexed by h such that $(g, h) \in \mathcal{L}$ of the $A(g, h)$. This shows that any open subset of G of the form gW is evenly covered by the projection from $\mathcal{L} \to G$. Hence, $\mathcal{L} \to G$ is a covering projection. Since G is simply connected, this implies that the projection $\mathcal{L} \to G$ is a diffeomorphism.

Claim 3.5. *If (g_1, h_1) and $(g_2, h_2) \in \mathcal{L}$, then so is $(g_1 g_2, h_1 h_2)$. That is to say $\mathcal{L}$ is a subgroup of $G \times H$ endowed with the product multiplication.*

Proof. Fix $(g_1, h_1) \in \mathcal{L}$ and consider the subset $X \subset \mathcal{L}$ consisting of all $(g_2, h_2) \in \mathcal{L}$ such that $(g_1 g_2, h_1 h_2) \in \mathcal{L}$. Since $(g_1 g_2, h_1 h_2) \in \mathcal{L}$, so is

$A(g_1g_2, h_1h_2)$ meaning for all $w \in W$, we have $(g_1g_2w, h_1h_2\varphi(w)) \in \mathcal{L}$. By definition $(g_2w, h_2\varphi(w))$ is a neighborhood of (g_2, h_2) in $\mathcal{L}$. This shows that X is an open subset of $\mathcal{L}$ in the leaf topology. Since the leaves are locallly closed smooth submanifolds, it is clear that X is a closed subset of $\mathcal{L}$. Obviously, if $(g_1, h_1) \in \mathcal{L}$ then so is (g_1e, h_1e), showing that $(e, e) \in X$. Since $\mathcal{L}$ is connected, it follows that $(g_1, h_1) \cdot \mathcal{L} \subset \mathcal{L}$ for every $(g_1, h_1) \in \mathcal{L}$; i..e., $\mathcal{L}$ is closed under multiplication in $G \times H$. $\square$

It follows that that the projection from $\mathcal{L} \to G$ is then an isomorphism of groups and hence of Lie groups.

Since $\mathcal{L}$ is a subgroup of $G \times H$, the projection to H gives a homomorphism $\mathcal{L} \to H$, which composed with the inverse of the projection of $\mathcal{L} \to G$ yields a Lie group homomorphism $G \to H$. Restricted to W it is the original map φ. In particular, the induced map on the Lie algebras is original Lie algebra map $\rho\colon \mathfrak{g} \to \mathfrak{h}$.

$\square$

Corollary 3.6. *Let L be a Lie algebra and G a simply connected Lie group and let $\rho\colon \mathfrak{g} \to L$ be an isomorphism of Lie algebras. If G' is a connected Lie group and we have an isomorphism of Lie algebras $\rho'\colon \mathfrak{g}' \to L$ then there is a homomorphism $f\colon G \to G'$ with $\rho' \circ D_e f = \rho$. Futhermore, kernel of f is a discrete subgroup contained in the center of G and f induces an isomorphism $G/\mathrm{Ker}(f) \to G'$.*

In particular, if G' is also simply connected then $f\colon G \to G'$ is an isomorphism of Lie groups compatible with the identifiactions of $\mathfrak{g}$ and $\mathfrak{g}'$ with L.

Proof. The previous result implies that there is a unique map $f\colon G \to G'$ with $\rho' \circ D_e f = \rho$. Since this map is an isomorphism on the Lie algebras, it induces an isomorphisms of suficiently small local Lie subgroups of G and G'. This implies that there is a neighborhood of the identity in G disjoint from the kernel of f. It follows that the kernel of f is a discrete subgroup. It is also a normal subgroup. Thus, the conjugation action of G on itself stabilizes $\mathrm{Ker}(f)$. But since this group is discrete and G is connected, this implies the conjufgation action of G on $\mathrm{Ker}(f)$ is trivial, meaning that $\mathrm{Ker}(f)$ is contained in the center of G.. $\square$

Corollary 3.7. *Given a simply connected Lie group G, then every connected Lie group with the same Lie algebra is isomorphic to G/Λ where Λ is a discrete subgroup of the center of G. Furthermore, the projection $G \to G/\Lambda$*

is a covering space with automorphism group Λ *acting by left multiplication on* G.