LIQUID REAL VECTOR SPACES: A SHORT INTRODUCTION

In this last talk we show why $\mathcal{M}$-complete vector spaces do not define an analytic ring structure on $\mathbb{R}$. The key feature is non-conventional functional analysis in the form of non-locally convex spaces. After this observation, we introduce the liquid analytic structures on $\mathbb{R}$. We will follow [CS20, Lectures V-VI].

1. Why $\mathcal{M}$-complete real vector spaces fail

Previously, we introduced a category of quasi-separated $\mathcal{M}$-complete real vector spaces. We showed that for V such a condensed real vector space, S a profinite map, and $f : S \to V$ a map of condensed sets, there is a unique extension to the space of Radon measures $\mathcal{M}(S, \mathbb{R}) \to V$; this even holds for S a compact Hausdorff space, and it is a condensed enhancement of the classical integration on locally convex vector spaces. However, if $\mathcal{M}$-complete real vector spaces (without the quasi-separated assumptions) form an analytic ring structure on $\mathbb{R}$, then $\mathcal{M}$-complete vector spaces should be stable under both extensions and cokernels. In the following section we will see that neither of these properties hold.

1.1. **Non stability under extensions and cokernels.** The non stability under extensions comes back to Ribet [Rib79] and his examples of extensions of locally convex spaces which are non locally convex.

Let $V = \ell^1(\mathbb{N})$ be the Banach space of ℓ^1-sequences of real numbers. The following lemma characterizes extensions of Banach spaces by $\mathbb{R}$.

Lemma 1.1 ([CS20, Lemma 5.2]). *Let V be a Banach space, let $V_0 \subset V$ be a dense subspace and let $\phi : V_0 \to \mathbb{R}$ be a function that is almost linear in the sense that there exists $C > 0$ such that for $v, w \in V_0$ we have*

(1) $\phi(av) = a\phi(v)$ *for* $a \in \mathbb{R}$ *and* $v \in V_0$.
(2)
$$|\phi(v + w) - \phi(v) - \phi(w)| \leq C(||v|| + ||w||).$$

Then $V_0' := \mathbb{R} \times V_0$ has topological vector space structure with a system of open neighbourhoods of 0 given by
$$\{(r, v) : ||v|| + |r - \phi(v)| < \varepsilon\}.$$
The completion V' of this vector spaces defines an extension $0 \to \mathbb{R} \to V' \to V \to 0$ of topological vector spaces (and so of condensed real vector spaces).

This extension is split (as condensed or topological vector spaces) if and only if there is a linear function $f_0 : V_0 \to \mathbb{R}$ such that $|f(v) - \phi(v)| \leq C'||v||$ for all $v \in V_0$ and some $C' > 0$.

Proof. The almost linear property of ϕ guarantee that the natural real vector space structure on $\mathbb{R} \times V_0$ is continuous, so it is indeed a topological real vector space. It is also clear that the completion V' provides such an extension. Note that two almost linear functions ϕ_1 and ϕ_2 that are equivalent in the sense that $|\phi_1(v) - \phi_2(v)| \leq C'||v||$ for all $v \in V_0$ and $C' > 0$ define the same topological vector space. Indeed, we have inequalities
$$|\phi_1(v + w) - \phi_1(v) - \phi_1(w)| \quad \leq |\phi_2(v + w) - \phi_2(v) - \phi_2(w)| + 2C'(||v|| + ||w||),$$
so if ϕ_2 is almost linear then so is ϕ_1, and
$$||v|| + |r - \phi_1(v)| \leq (1 + C')||v|| + |r - \phi_2(v)|,$$
so both ϕ_1 and ϕ_2 define cofinal system of neighbourhoods of 0. The map ϕ completes to a (not necessarily linear) continuous map of topological spaces $\phi : V \to V'$.

Thus, if $\phi = f$ is linear, then $V' = V \oplus \mathbb{R}$. Conversely, suppose that V' is split, then we can find a section $s : V \to V'$ of topological real vector spaces. Restricting to V_0 we find a linear map $f : V_0 \to \mathbb{R}$ such that $s(v) = (f(v), v)$ for $v \in V_0$. Now, the map s being continuous precisely means that $|f(v) - \phi(v)| \leq C'||v||$ for some $C' > 0$, proving the claim. $\square$

From the previous lemma we see that in order to construct a non-split extension we need to find a function that is almost linear but not equivalent to a linear function. An example of such a function is given by entropy: for $p_1, \dots, p_n$ numbers in $[0,1]$ with sum 1 their entropy is

$$H = -\sum_{i=1}^{n} p_i \log p_i.$$

The almost linearity property of $x \log |x|$ follows by the next lemma.

Lemma 1.2 ([CS20, Lemma 5.3]). *For all real number s and t, one has*

$$|s \log |s| + t \log |t| - (s+t) \log |s+t|| \leq (2 \log 2)(|s| + |t|).$$

Proof. Both terms are homogeneous for the multiplication by $\lambda \in \mathbb{R}$ in s and t. Thus, without loss of generality we can assume that $t = 1$ and $s \in [-1, 1]$. We need to show that

$$|s \log |s| - (s+1) \log |s+1|| \leq (2 \log 2)(|s| + 1).$$

For this, it suffices to show that the left hand side is bounded by $2 \log 2$, which is an easy calculus exercise. $\square$

Corollary 1.3. *Let $V_0 \subset V = \ell^1(\mathbb{N})$ be the subspace spanned by sequences with finite support. The function*

$$H : (x_0, x_1, \dots) \in V_0 \mapsto s \log |s| - \sum_{i \geq 0} x_i \log |x_i|, \text{ where } s = \sum_{i \geq 0} x_i,$$

is locally almost linear but not globally almost linear, and so defines a non-split extension

$$0 \to \mathbb{R} \to V' \to V \to 0.$$

Proof. Almost linearity follows by Lemma 1.2. For global non linearity, suppose that H is close to $\sum_i \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$. Looking at the basis $e_n = (\cdots, 0, 0, 1, 0, 0, \cdots)$, one sees that the λ_i are bounded (as $H = 0$ on such points). On the other hand, looking at $f_n = \sum_{i=0}^{n-1} \frac{1}{n} e_i = (\frac{1}{n}, \dots, \frac{1}{n}, 0 \dots)$, global almost linearity produces

$$|H(n) - \frac{1}{n} \sum_{i=0}^{n-1} \lambda_i| \leq C$$

for some constant C. This would require $H(n)$ to be bounded, which is not true since $H(n) = \log(n)$. $\square$

Let $W_1 := \mathcal{M}(\mathbb{N} \cup \{\infty\}, \mathbb{R})/(\infty)$. We can construct the counter example explicitly as follows.

Definition 1.4. Consider the following increasing union of compact Hausdorff spaces

$$\widetilde{W_1} = \bigsqcup_{c>0} \{(x_0, x_1, \dots, y_0, y_1, \dots) \in \prod_{\mathbb{N}} [-c, c] \times \prod_{\mathbb{N}} [-c, c] | \sum_n (|x_n| + |y_n - x_n \log |x_n||) \leq c\}.$$

Proposition 1.5 ([CS20, Proposition 5.6]). *The condensed set $\widetilde{W_1}$ has a natural structure of condensed real vector space and sits in a exact sequence*

$$0 \to W_1 \to \widetilde{W_1} \to W_1 \to 0$$

given by $(y_0, y_1, \dots) \mapsto (0, \dots, y_0, y_1, \dots)$ and $(x_0, x_1, \dots, y_1, y_2, \dots) \mapsto (x_0, x_1, \dots)$.

Proof. The structure of condensed real vector space follows by Lemma 1.2. The sequence is clearly left exact, the map $\widetilde{W_1} \to W_1$ is surjective by taking $y_n = x_n \log |x_n|$. $\square$

Proposition 1.6. *The space $\widetilde{W_1}$ is not $\mathcal{M}$-complete.*

Proof. The sequence e_n with $x_n = 1$ and all other terms zero is a null-sequence in $\widetilde{W_1}$ (open neighbourhood of 0 in the product if intervals contains all but finitely many copies). Then, if it was $\mathcal{M}$-complete we could construct a map $\mathcal{M}(\mathbb{N} \cup \{\infty\}, \mathbb{R}) \to \widetilde{W}$ vanishing at infinity, so a section $W \to \widetilde{W}$. But we reconstruct the extension $0 \to \mathbb{R} \to V' \to \ell^1(\mathbb{N}) \to$ from $0 \to W_1 \to \widetilde{W_1} \to W_1 \to 0$ as follows: first take a pullback along the map $\ell^1(\mathbb{N}) \to W_1$, the new extension $0 \to W_1 \to W' \to \ell^1(\mathbb{N}) \to 0$ is actually the pushout along $\ell^1(\mathbb{N}) \to W_1$ of another short exact sequence $0 \to \ell^1(\mathbb{N}) \to V'' \to \ell^1(\mathbb{N}) \to 0$, then the extension V' is construct by the sum of all x_i's $\ell^1(\mathbb{N}) \to \mathbb{R}$. Thus a split of $\widetilde{W_1} \to W_1$ would produce a split of $V' \to \ell^1(\mathbb{N})$ (after carefully following the steps before) which is not possible by Corollary 1.3. $\square$

We now provide an example of a cokernel of $\mathcal{M}$-complete real vector spaces which is not $\mathcal{M}$-complete. Note however that any such space must be non quasi-separated. Let $W_\infty = \bigcup_{c>0} \prod_{\mathbb{N}} [-c, c]$ be the Smith space of bounded null sequences, we have an inclusion $W_1 \subset W_\infty$.

Proposition 1.7 ([CS20, Proposition 5.8]). *The quotient W_∞/W_1 is not $\mathcal{M}$-complete. more precisely, the map $f : W_1 \to W_\infty$ of condensed sets given by $f(x_0, x_1, \ldots) = (x_0 \log |x_0|, x_1 \log |x_1|, \ldots)$ induces a non-zero map of condensed real vector spaces $W_1 \to W_\infty/W_1$ whose restriction to the null-sequence e_n vanishes.*

Proof. In contrast to what happened in Proposition 1.6, what fails in this situation is not the extension condition to Radon measures, but the uniqueness of such.

We first show that the map $f : W_1 \to W_\infty/W_1$ is a map of condensed abelian groups. This is equivalent to show that a precise function $W_1 \times W_1 \to W_\infty/W_1$ vanishes. It is the projection along $W_\infty \to W_\infty/W_1$ of the map $W_1 \times W_1 \to W_\infty$ given by sending $(x_0, x_1, \ldots,) \times (y_0, y_1, \ldots)$ to $(x_0 \log |x_0| + y_0 \log |y_0| - (x_0 + y_0) \log |x_0 + y_0|, (x_1 \log |x_1| + y_1 \log |y_1| - (x_1 + y_1) \log |x_1 + y_1|, \ldots)$ which lies in W_1 by Lemma 1.2, this gives additivity. For $\mathbb{R}$-linearity, we consider instead the map $\mathbb{R} \times W_1 \to W_\infty$ given by mapping $(r) \times (x_0, \ldots)$ to $(rx_0 \log |rx_0| - rx_0 \log |x_0|, rx_1 \log |rx_1| - rx_1 \log |x_1|, \ldots)$, but $rx_0 \log |rx_0| - rx_0 \log |x_0| = rx_0 \log r$, proving that it lands in W_1.

To see that the map f is non-zero it suffices to see that the image of W_1 in W_∞ does not land in W_1, but the sequence $(\frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots)$ with n occurrences of n defines a map $\mathbb{N} \cup \{\infty\} \to W_1$ whose image under f is the sequence $(-\frac{\log n}{n}, \ldots, -\frac{\log n}{n}, 0, \ldots)$ which does not have bounded ℓ^1-norm. □

2. Liquid real numbers

One of the problems why $\mathcal{M}$-complete condensed real vector spaces do not form an analytic ring is that there are extensions of locally convex vector spaces which are non locally convex. Let us recall the following definition

Definition 2.1 ([CS20, Definition 6.1]). For $0 < p \leq 1$, a p-Banach space is a topological $\mathbb{R}$-vector space V such that there exists a p-norm, i.e. a continuous map

$$|| \cdot || : V \to \mathbb{R}_{\geq 0}$$

with the following properties:

(1) For any $v \in V$, $||v|| = 0$ if and only if $v = 0$.
(2) For any $v \in V$ and $a \in \mathbb{R}$ we have $||av|| = |a|^p ||v||$.
(3) For any $v, w \in V$ we have $||v + w|| \leq ||v|| + ||w||$.
(4) The sets $\{v : ||v|| \leq \varepsilon\}$ form a basis of neighbourhoods of 0.
(5) For any sequence $v_0, v_1, \ldots \in V$ such that $||v_i - v_j|| \to 0$ as $i, j \to \infty$ there is a unique $v \in V$ with $||v - v_i|| \to 0$.

A quasi-Banach space is a p-Banach space for some $0 < p \leq 1$.

The following theorem explains the behaviour of extensions of quasi-Banach spaces.

Theorem 2.2 ([Kal81]). *An extension of two p-Banach spaces is p'-Banach for all $p' < p$.*

The previous theorem motivates the following definition of p-Radon measures.

Definition 2.3 ([CS20, Definition 6.3]). Let $S = \varprojlim_i S_i$ be a profinite set written as a limit of finite sets. For $0 < p < 1$ we define

$$\mathcal{M}_p(S, \mathbb{R}) = \bigsqcup_{c>0} \mathcal{M}_p(S, \mathbb{R})_{\leq c}$$

where

$$\mathcal{M}_p(S, \mathbb{R})_{\leq c} = \varprojlim_i \mathbb{R}[S_i]_{\ell^p \leq c}$$

and $\mathbb{R}[S_i]_{\ell^p \leq c}$ is the spaces of sequences $(a_s)_{s \in S_i}$ such that $\sum_{s \in S_i} |a_s|^p \leq c$. Equivalently, $\mathcal{M}_p(S, \mathbb{R})$ is the space of functions $\mu : Open(S) \to \mathbb{R}$ satisfying the following properties:

(1) For any finite disjoint union $\bigsqcup_i U_i \subset S$ of open subspaces of S we have

$$\mu(\bigsqcup_i U_i) = \sum_i \mu(U_i).$$

(2) There is $C > 0$ such that for any disjoint union $S = \bigsqcup_{i=1}^n U_i$ we have

$$\sum_{i=1^n} |\mu(U_i)|^p \leq C.$$

For $0 < p \leq 1$ we define

$$\mathcal{M}_{<p}(S, \mathbb{R}) = \bigcup_{p' < p} \mathcal{M}_{p'}(S, \mathbb{R}).$$

The idea behind the spaces of $(< p)$-Radon measures is that they capture the non-locally convex behaviour that extensions of Banach spaces satisfy by Theorem 2.2. The first property that the spaces of Radon measures must satisfy is that they are functorial for maps between profinite sets, this follows from the fact that any map of finite sets $S \to T$ gives rise a map $\mathbb{R}[S]_{\ell^p \leq c} \to \mathbb{R}[T]_{\ell^p \leq c}$.

Let us now state the main theorem regarding the liquid analytic structures.

Theorem 2.4 ([CS20, Theorems 6.5 and 6.6])**.** *Let $0 < p \leq 1$, the datum $(\mathbb{R}, S \mapsto \mathcal{M}_{<p}(S, \mathbb{R}))$ defines an analytic ring structure on $\mathbb{R}$. More precisely, let $\mathrm{Liq}_{<p}(\mathbb{R})$ be the category of condensed real vector spaces V such that for any extremally disconnected set S, any map $S \to V$ extends uniquely to $\mathcal{M}_{<p}(S, \mathbb{R}) \to V$. The following holds:*

(1) $\mathrm{Liq}_{<p}(\mathbb{R})$ *is an abelian category stable under limits, colimits and extensions of* $\mathrm{Cond}(\mathbb{R})$.
(2) *The inclusion* $\mathrm{Liq}_{<p}(\mathbb{R}) \to \mathrm{Cond}(\mathbb{R})$ *has a left adjoint* $(-)^{\mathrm{liq}_{<p}}$ *which is the unique colimit preserving functor mapping* $\mathbb{R}[S]$ *to* $\mathcal{M}_{<p}(S, \mathbb{R})$ *for* S *extremally disconnected.*
(3) *The category* $\mathrm{Liq}_{<p}(\mathbb{R})$ *is symmetric monoidal with tensor* $- \otimes_{<p} -$ *and* $(-)^{\mathrm{liq}_{<p}}$ *is symmetric monoidal.*

Moreover, let $\mathscr{C} \subset \mathscr{D}(\mathrm{Cond}(\mathbb{R}))$ be the full subcategory of complexes M such that for all S extremally disconnected the following map is an equivalence:

$$R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}_{<p}(S, \mathbb{R}), M) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathbb{R}}(\mathbb{R}[S], M).$$

The following hold:

(1) *The category $\mathscr{C}$ is stable under all limits, colimits and Postnikov towers in $\mathscr{D}(\mathrm{Cond}(\mathbb{R}))$.*
(2) *An object $M \in \mathscr{D}(\mathrm{Cond}(\mathbb{R}))$ is in $\mathscr{C}$ if and only if $H^i(M) \in \mathrm{Liq}_{<p}(\mathbb{R})$ for all $i \in \mathbb{Z}$.*
(3) *The natural functor $\mathscr{D}(\mathrm{Liq}_{<p}(\mathbb{R})) \to \mathscr{D}(\mathrm{Cond}(\mathbb{R}))$ is fully faithful and has $\mathscr{C}$ by essential image.*
(4) *The forgetful functor $\mathscr{D}(\mathrm{Liq}_{<p}(\mathbb{R})) \to \mathscr{D}(\mathrm{Cond}(\mathbb{R}))$ has a left adjoint $(-)^{L\,\mathrm{liq}_{<p}}$ which is the left derived functor of $(-)^{\mathrm{liq}_{<p}}$. Moreover $\mathscr{D}(\mathrm{Liq}_{<p}(\mathbb{R}))$ is symmetric monoidal with tensor $- \otimes^L_{<p} -$ and $(-)^{L\,\mathrm{liq}_{<p}}$ is a symmetric monoidal functor.*
(5) *The tensor $- \otimes^L_{<p} -$ is the left derived functor of $- \otimes_{<p} -$.*

To provide a hint of what liquid real vector spaces look like, let us study the quasi-separated ones. We need a definition.

Definition 2.5. Let $0 < q \leq 1$. A quasi-compact subobject K of a condensed real vector space V is said q-convex if for all finite sets S, the map

$$\mathbb{R}^S_{\ell^q \leq 1} \times K^S \to V,$$
$$((\lambda_s), (x_s)) \mapsto \sum_s \lambda_s x_s$$

lands inside K.

The following theorem is analogue to the one we proved for $\mathcal{M}$-complete real vector spaces.

Theorem 2.6 ([CS22, Theorem 2.14])**.** *Let V be a quasi-separated condesed real vector space and $0 < p \leq 1$. Then V is p-liquid if and only of for eqvery $q < p$, every quasi-compact subspace $K \subset V$ is contained in a quasi-compact q-convex subspace of V. In particular, any p-Banach space (and so any complete p-locally convex vector space) is p-liquid.*

Proof. Suppose that V is liquid, and let $K \subset V$ a quasi-compact subspace. Let us choose a surjection $S \to K$ from an extremally disconnected set. We have an extension $\mathcal{M}_{<p}(S, \mathbb{R}) \to V$. Let us consider the restriction $\mathcal{M}_q(S, \mathbb{R})_{\leq 1} \to V$ and let Q denote its image. Then Q is a quasi-compact subspace of V that is q-convex as $\mathcal{M}_q(S, \mathbb{R})_{\leq 1}$ is a compact q-convex subspace of $\mathcal{M}_q(S, \mathbb{R})$.

Conversely, suppose that V satisfies the hypothesis of the theorem, let S be an extremally disconnected set and $S \to V$ a map of condensed sets. The free condensed real vector space $\mathbb{R}[S] \subset \mathcal{M}_{<p}(S, \mathbb{R})$ is dense, so it suffices to prove the existence of the map $\mathcal{M}_{<p}(S, \mathbb{R}) \to V$. Moreover, it suffices to show the existence of the map $\mathcal{M}_q(S, \mathbb{R}) \to V$ for all $q < p$. Let Q be the image of S in V and let Q' be a q-convex quasi-compact subspace of V containing Q. The $\mathbb{R}$-span $W = \bigcup_{c>0} cQ' \subset V$ is a condensed real vector space (for stability under addition note that for $v, w \in Q'$ the object $\frac{1}{n}v + \frac{1}{n}w$ remains in Q' for $n >> 0$). It suffices to show that there is a unique lift $\mathcal{M}_q(S, \mathbb{R}) \to W$. Let us write $S = \varprojlim_i S_i$ as a limit of finite sets with projections $\pi_i : S \to S_i$, and let $\gamma_i : S_i \to S$ be any lift. Let $\mu \in \mathcal{M}_q(S, \mathbb{R})_{\leq 1}$ and consider the net

$$\{\sum_{s \in S_i} f(\gamma_i(s))\mu(\pi_i^{-1}(s))\}_{i,\gamma_i} \in W. \tag{2.1}$$

The space W is q-locally convex, and the same argument for $\mathcal{M}$-complete vector spaces shows that (2.1) is a Cauchy net, whose limit is independent of the lift $\gamma_i : S_i \to S$. Note that in the argument we use that $\sum_{s \in S_i} |\mu(\pi_i^{-1}(s))|^q \leq 1$ to guarantee that if $f(\gamma_i(s))$ lands in a q-convex neighbourhood U of 0, then so is the sum $\sum_{s \in S_i} f(\gamma_i(s))\mu(\pi_i^{-1}(s))$. □

References

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`, 2020.

[CS22] Dustin Clausen and Peter Scholze. Condensed Mathematics and Complex Geometry. `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`, 2022.

[Kal81] N. J. Kalton. Convexity, type and the three space problem. *Stud. Math.*, 69:247–287, 1981.

[Rib79] M. Ribe. Examples for the nonlocally convex three space problem. *Proc. Am. Math. Soc.*, 73:351–355, 1979.