CONDENSED REAL VECTOR SPACES

So far we have studied the non-archimedean picture of condensed mathematics in the form of solid abelian groups. In the last lectures of the seminar we will scratch the archimedean theory, first in the form of classical (locally convex) analysis, and then with the introduction of the liquid theory and non-conventional (locally concave) analysis.

1. Locally convex vector spaces in condensed math

Let V be a Banach space with norm $||-||$, it is Hausdorff as topological space, so it defines a condensed set $\underline{V}$ with values at S extremally disconnected

$$\underline{V}(S) = C(S, V).$$

Since the condensification functor is a right adjoint, the structure of real topological vector space of V transfers to an structure of condensed $\mathbb{R}$-module on $\underline{V}$. Moreover, since V is metrizable, it is compactly generated, and the underlying topological space of $\underline{V}$ is naturally isomorphic to V. This gives rise a fully-faithful embedding fro the category of real Banach spaces into the category of real condensed vector spaces.

More generally, let V be a complete locally convex real vector space. By (an equivalent) definition, V is a real topological vector space whose topology is determined with respect to a family of seminorms, for which V is Hausdorff and complete. By taking the completions of those norms, we can write V as a cofiltered limit $V = \varprojlim_i V_i$ of real Banach spaces V_i. Then, the associated condensed set of V is nothing but

$$\underline{V} = \varprojlim_i \underline{V}_i.$$

Moreover, if V is compactly generated, $V(*)_{top}$ is naturally isomorphic to V. We have shown the following (formal) lemma

Lemma 1.1. *Let* $\mathrm{LocCon}_{\mathbb{R}}$ *be the category of complete locally convex real vector spaces, the condensification functor defines a functor*

$$\underline{(-)} : \mathrm{LocCon}_{\mathbb{R}} \to \mathrm{Mod}_{\mathbb{R}}(\mathrm{CondAb})$$

taking values in condensed real vector spaces. Moreover, when restricted to compactly generated complete locally convex real vector spaces, the functor is fully faithful.

Banach spaces also appear from mapping spaces from compact Hausdorff spaces:

Lemma 1.2. *Let V be a Banach space and K a compact Hausdorff space, then*

$$\underline{C(K,V)} = \underline{\mathrm{Hom}}(\mathbb{Z}[K], \underline{V})$$

is the condensification of the Banach space $C(K,V)$ endowed with the sup norm.

Proof. The evaluation map $C(K,V) \times K \to V$ is continuous, this gives rise a map of condensed sets $\underline{C(K,V)} \times K \to \underline{V}$, and so a morphism

$$\underline{C(K,V)} \to \underline{\mathrm{Hom}}(\mathbb{Z}[K], \underline{V}),$$

we want to prove it is an isomorphism, but this follows from the fact that for a profinite set S a the datum of a continuous map $S \to C(K,V)$ is equivalent to a continuous map $S \times K \to V$, proving what we wanted. $\square$

2. Radon measures and Smith spaces

In the construction of the analytic ring $\mathbb{Z}_{\blacksquare}$, we interpreted the free solid abelian groups $\mathbb{Z}_{\blacksquare}[S]$ for S profinite, as a space of $\mathbb{Z}$-valued Radon measures

$$\mathbb{Z}_{\blacksquare}[S] = \underline{\mathrm{Hom}}_{\mathbb{Z}}(C(S,\mathbb{Z}),\mathbb{Z}).$$

One might hope that the space of real Radon measures $\mathcal{M}(S,\mathbb{R})$ might define an analytic ring structure on $\mathbb{R}$ which encodes classical functional analysis. This approach is not going to work, but still leads to an interesting notion of $\mathcal{M}$-complete real condensed vector spaces. Before going into this, let us recap some properties of Radon measures from the perspective of condensed math. First, we give an equivalent definition of signed Radon measures for profinite sets:

Definition 2.1. Let $S = \varprojlim_i S_i$ be a profinite set, the space of signed real valued Radon measures of S is the condensed real vector space

$$\mathcal{M}(S,\mathbb{R}) = \bigcup_{c>0} \varprojlim_i \mathbb{R}[S_i]_{\ell^1\leq c},$$

where $\mathbb{R}[S_i]_{\ell^1\leq c}$ is the compact subspace of $\mathbb{R}[S_i]$ of objects $\sum_{s\in S_i} a_s s$ such that $\sum_{s\in S_i}|a_s| \leq c$. Equivalently, $\mathcal{M}(S,\mathbb{R})$ is the space of functions $\mu : \mathrm{Open}(S)\to\mathbb{R}$ endowed with the weak topology, satisfying the following properties:

1. For any disjoint union of open subspaces $\bigsqcup_{i=1}^k U_i \subset S$ we have
$$\mu(\bigsqcup_{i=1}^k U_i) = \sum_{i=1}^k \mu(U_i).$$
2. There is some $c\geq 0$ such that for any any disjoint open cover $\bigsqcup_{i=1}^k U_i = S$ we have
$$\sum_{i=1}^k |\mu(U_i)| \leq c.$$

The previous definition of Radon measures is much simpler that the classical one using the Borel σ-algebra. Even better, for K a compact Hausdorff space, the space of signed Radon measures $\mathcal{M}(K,\mathbb{R})$ endowed with the weak topology, i.e. the subspace topology of the embedding

$$\mathcal{M}(K,\mathbb{R}) \subset \prod_{U\subset K}\mathbb{R}$$

obtained by evaluating a measure μ at open subspaces of K, is constructed by descent from the spaces $\mathcal{M}(S,\mathbb{R})$ from S profinite. More precisely, if S is profinite and $f: S\to K$ is a surjection, we have a right exact sequence of condensed real vector spaces

$$\mathcal{M}(S\times_K S,\mathbb{R}) \xrightarrow{p_{1,*}-p_{2,*}} \mathcal{M}(S,\mathbb{R}) \xrightarrow{f_*} \mathcal{M}(K,\mathbb{R}) \to 0.$$

This can be proved explicitly following the definitions, but is also a consequence of Smith's duality theorem, see §4.

By construction, the spaces of Radon measures $W=\mathcal{M}(S,\mathbb{R})$ have a convex compact Hausdorff subspace $K=\varprojlim_i \mathbb{R}[S_i]_{\ell^1\leq 1}\subset W$ such that $W=\bigcup_{c>0} cK$ as condensed sets. Note that the spaces cK are not open in W, and that W does not have the inductive limit topology. However, W is still the condensed space associated to a locally convex real vector space. We can make this property a definition:

Definition 2.2. A Smith space space W is a locally convex real/condensed vector space such that there is a convex compact subspace $K\subset W$ with $W=\bigcup_c cK$.

3. $\mathcal{M}$-complete real vector spaces

We can define the category of $\mathcal{M}$-complete real vector spaces, heuristically speaking it is the category of condensed real vector spaces for which maps from profinite sets can be *naturally integrated* with respect to Radon measures.

Definition 3.1. A condensed real vector space V is $\mathcal{M}$ complete if for any profinite set S and map $S \to V$, there is a unique extension of condensed real vector spaces

$$\mathcal{M}(S, \mathbb{R}) \to V.$$

Remark 3.2. Let V be a quasi-separated condensed real vector space, since the Dirac measures $S \subset \mathcal{M}(S, \mathbb{R})$ span a dense subspace, there is at most one extension of a map $f : S \to V$ to $\mathcal{M}(S, \mathbb{R}) \to V$. Then, the condition for V being an $\mathcal{M}$-complete real vector space is just about extending f.

The first class of spaces that we should check to be $\mathcal{M}$-complete is the category of complete locally convex real vector spaces, this is a reflection of the fact that we can integrate V-valued functions from compact Hausdorff spaces.

Proposition 3.3. *Let S be a profinite set, V a complete locally convex real vector space and $f : S \to V$ a map of condensed sets. Then there is a unique extension to condensed real vector spaces*

$$\mathcal{M}(S, \mathbb{R}) \to V.$$

In other words, V is $\mathcal{M}$-complete.

Proof. It suffices to show that there is a natural extension $\mathcal{M}(S, \mathbb{R}) \to V$ as topological vector space. Let us write $S = \varprojlim_{i \in I} S_i$ as a limit of finite sets. For each i let us pick a section $g_i : S_i \to S$ (without any compatibility hypothesis). Let $\mu \in \mathcal{M}(S, \mathbb{R})$, consider the net $\mathscr{S} = \{\sum_{s \in S_i} f(g_i(s))\mu(\pi_i^{-1}(s))\}$, we claim that it is a Cauchy net. Indeed, let $U \subset V$ be a convex open subspace, since $f : S \to V$ is continuous, we can suppose that for all i large enough, and any two lifts $g_i : S_i \to S$ and $h_i : S_i \to S$, the difference $f(g_i(s)) - f(h_i(s))$ is in U for all $s \in S_i$. Thefore, for all $j \geq i$ large enough, the difference $f(g_i(\pi_i(s))) - f(g_j(s))$ is in U for $s \in S_j$. One deduces that $\mathscr{S}$ is a Cauchy net by using (1) and (2) of Definition 2.1. Thus, the net converges to a limit point $\int_S f d\mu$ (independent of the lifts of S_i to S). Linearity and continuity of the map $\mathcal{M}(S, \mathbb{R}) \to V$ follow by similar arguments. $\square$

As a corollary, we deduce that complete locally convex vector spaces are built up from Smith spaces when considered as a condensed real vector space.

Corollary 3.4. *Let V be a complete locally convex real vector space. Then V is the filtered colimit of its Smith subspaces when considered as a condensed real vector space.*

Proof. Let S be a profinite set and let $f : S \to \underline{V}$ be a continuous map. By Proposition 3.3 f extends to $\mathcal{M}(S, \mathbb{R})$. Let K be the image of $\mathcal{M}(S, \mathbb{R})_{\leq 1}$, then K is a convex qcqs subspace of V, so a compact Hausdorff subspace. Taking $W = \bigcup_{c>0} cK$ we see that $W \subset V$ is a Smith subspace, this shows that V is the union of its Smith subspaces. Let now $W_i = \bigcup_{c>0} cK_i$ for $i = 1, 2$ be two Smith subspaces of V, and let $K = K_1 \cup K_2 \subset V$. Let S be an extremally disconnected with a surjection $S \to K$, then the composite $S \to K \to V$ extends to $\mathcal{M}(S, \mathbb{R}) \to V$ and the image K_3 of $\mathcal{M}(S, \mathbb{R})_{\leq 1}$ is a convex compact Hausdorff subspace of V containing K_1 and K_2 respectively. Taking $W_3 = \bigcup_{c>0} cK_3$ we see that $W_1, W_2 \subset W_3$, proving that the category of Smith subspaces of V is filtered. $\square$

We can then characterize quasi-separated $\mathcal{M}$-complete condensed real vector spaces:

Theorem 3.5. *Let V be a $\mathcal{M}$-complete real vector space, then V is the filtered colimit of its Smith subspaces. Moreover, a filtered colimit of Smith spaces under injective transition maps is $\mathcal{M}$-complete. Let $f : V \to V'$ be a linear map of $\mathcal{M}$-complete real vector spaces, then the kernel and the image of f is $\mathcal{M}$-complete. Moreover, f admits a cokernel in the category of $\mathcal{M}$-complete quasi-separated real vector spaces, given by the quasi-separation of the cokernel.*

Proof. Let V be an $\mathcal{M}$-complete space, the same argument as in Corollary 3.4 shows that the category of Smith subspaces of V is filtered and that V is the union of its Smith subspaces. Conversely, suppose that $V = \varinjlim_i W_i$ is a filtered colimit of Smith subspaces under injective transition maps. Let S be profinite and consider $g : S \to V$, since S is quasi-compact, there is i such that g factors through $S \to W_i$, but W_i is locally convex, so we have an extension $\mathcal{M}(S, \mathbb{R}) \to W_i \to V$ proving that V is $\mathcal{M}$-complete.

Finally, it is clear the the kernel and image of the map f is a filtered union of Smith spaces, namely, any quasi-separated quotient of a Smith space is a Smith space. This also shows that the quasi-separation of the cokernel of f is a filtered union of Smith subspaces, so $\mathcal{M}$-complete. $\square$

4. Smith's theorem: duality between Banach and Smith spaces

We go back to the duality between Banach and Smith spaces in condensed real groups.

Theorem 4.1. *The functor $V \mapsto \underline{\mathrm{Hom}}_{\mathbb{R}}(V, \mathbb{R})$ induces an anti-equivalence between the categories of Banach and Smith real vector spaces.*

Proof. We first show that for S a profinite set there are natural isomorphisms
$$C(S, \mathbb{R}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{R}}(\mathcal{M}(S, \mathbb{R}), \mathbb{R}) \tag{4.1}$$
and
$$\mathcal{M}(S, \mathbb{R}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathbb{R}}(C(S, \mathbb{R}), \mathbb{R}). \tag{4.2}$$
By Proposition 3.3 we have a bilinear pairing
$$\mathcal{M}(S, \mathbb{R}) \times C(S, \mathbb{R}) \to \mathbb{R}.$$
This map is continuous, namely, the construction by Cauchy nets of the proposition shows that if $f : S \to \mathbb{R}$ has norm $||f|| < \varepsilon$, and $\mu \in \mathcal{M}(S, \mathbb{R})_{\leq 1}$, then $|\int_S f d\mu| < \varepsilon$. This pairing give rise natural maps of condensed real vector spaces as in (4.1) and (4.2).

We prove that (4.1) is an equivalence, we need to show that for all profinite set T the map
$$C(T, C(S, \mathbb{R})) \to \mathrm{Hom}_{\mathbb{R}}(\mathcal{M}(S, \mathbb{R}), C(T, \mathbb{R}))$$
is a bijection, this follows from the fact that $C(T, \mathbb{R})$ is $\mathcal{M}$-complete.

We now prove that (4.2) is an equivalence. We need to show that for any profinite set T the map
$$C(T, \mathcal{M}(S, \mathbb{R})) \to \mathrm{Hom}_{\mathbb{R}}(C(S, \mathbb{R}), C(T, \mathbb{R}))$$
is a bijection. After reescaling it suffices to show that $C(T, \mathcal{M}(S, \mathbb{R})_{\leq 1})$ is in bijection with the space of continuous maps $f : C(S, \mathbb{R}) \to C(T, \mathbb{R})$ of sup norm ≤ 1. Let us write $S = \varprojlim_i S_i$ as a limit of finite sets, since $\varinjlim_i C(S_i, \mathbb{R})$ is dense and isometrically embedded into $C(S, \mathbb{R})$, maps f of norm ≤ 1 as before are the same as a compatible system of maps of maps $f_i : C(S_i, \mathbb{R}) \to C(T, \mathbb{R})$ uniformly bounded by 1. It is clear that the datum of such f_i is the same as an object in $\mu_i \in C(T, \mathcal{M}(S_i, \mathbb{R})_{\leq 1})$, and so f arises from the limit of measures $\mu = \varprojlim_i \mu_i \in \varprojlim_i C(T, \mathcal{M}(S_i, \mathbb{R})_{\leq 1}) = C(T, \mathcal{M}(S, \mathbb{R})_{\leq 1})$. This proves what we wanted.

We now deal with general Smith and Banach spaces. Let $W = \bigcup_{c>0} cK$ be a Smith space with K a convex compact subspace of W. Let S be profinite and let $S \to K$ be a surjection. Since W is $\mathcal{M}$-complete, we get a map of Smith spaces
$$f : \mathcal{M}(S, \mathbb{R}) \to W$$
that is necessarily a surjection of condensed real vector spaces. The kernel of f is necessarily another Smith space, namely, it is of the form
$$\ker f = \bigcup_{c>0} c(\mathcal{M}(S, \mathbb{R})_{\leq 1} \cap \ker f),$$
by repeating the same construction we find a right exact sequence
$$\mathcal{M}(T, \mathbb{R}) \to \mathcal{M}(S, \mathbb{R}) \to W. \tag{4.3}$$
Taking duals on (4.3) we get a left exact sequence
$$0 \to W^\vee \to C(S, \mathbb{R}) \to C(T, \mathbb{R}),$$
proving that $W^\vee$ is a Banach space.

Conversely, let V be a Banach space equipped with a norm $|| - ||$. The ball $B = \mathrm{Hom}^0_{\mathbb{R}}(V, \mathbb{R})_{||-|| \leq 1}$ is a compact Hausdorff space when endowed with the weak topology: this is known as the Banach-Alaoglu's theorem, it follows from the fact that B is embedded as a closed subspace $B \to \prod_{\substack{v \in V \\ ||v|| \leq 1}} [-1, 1]$ and Tychonoff's theorem. Taking a surjection from a profinite set $S \to B$ we get a map $V \to C(S, \mathbb{R})$ that is a closed immersion, indeed it is even an isometric inclusion by Hahn-Banach theorem. Repeating the process we find a left exact sequence
$$0 \to V \to C(S, \mathbb{R}) \to C(S', \mathbb{R})$$
with S and T profinite sets. We claim that taking $\underline{\mathrm{Hom}}_{\mathbb{R}}(-, \mathbb{R})$ gives rise a right exact sequence
$$\mathcal{M}(S', \mathbb{R}) \to \mathcal{M}(S, \mathbb{R}) \to V^\vee \to 0.$$

For this, we have to show that for all extremally disconnected T we have a right exact sequence

$$\mathrm{Hom}_{\mathbb{R}}(C(S',\mathbb{R}),C(T,\mathbb{R}))\to \mathrm{Hom}_{\mathbb{R}}(C(S,\mathbb{R}),C(T,\mathbb{R}))\to \mathrm{Hom}_{\mathbb{R}}(V,C(T,\mathbb{R}))\to 0.$$

In other words, we need to see that $C(T,\mathbb{R})$ is injective in the category of Banach spaces for T extremally disconnected. In other words, we need to show that for any closed immersion of Banach spaces $V'\subset V$, any map $V'\to C(T,\mathbb{R})$ extends to a map $V\to C(T,\mathbb{R})$.

Since T is extremally disconnected, it is a retract of a Stone-Čech compactification βS_0 of a discrete set S_0, so it suffices to show that $C(\beta S_0,\mathbb{R})=\ell^{\infty}(S_0,\mathbb{R})$ is injective. But then, taking a continuous map $f:V'\to \ell^{\infty}(S_0,\setminus)$ and its projections $f_s:V'\to \mathbb{R}|_s$ for $s\in S_0$, the Hahn-Banach theorem provides an extension $F_s:V'\to\mathbb{R}|_s$ of norm $||F_s||=||f_s||$. Then, the collection of functions F_s defines an extension $F:V\to \ell^{\infty}(S_0,\mathbb{R})$ of f, proving the injectivity of $\ell^{\infty}(S_0,\mathbb{R})$.

We have shown that for a left exact sequence of Banach spaces

$$0\to V\to C(S,\mathbb{R})\to C(S',\mathbb{R})$$

the dual is a right exact sequence of Smith spaces, and that for a right exact sequence of Smith spaces

$$C(S',\mathbb{R})\to C(S,\mathbb{R})\to W\to 0,$$

the dual is a left exact sequence of Banach spaces. Furthermore, we know that $C(S,\mathbb{R})$ are $\mathcal{M}(S,\mathbb{R})$ are dual each other for S profinite. One formally deduces that duality produces an anti-equivalence between Smith and Banach spaces. ☐

Corollary 4.2. *Let K be a compact Hausdorff space and let $S\to K$ be a surjection from a profinite set. Then we have a right exact sequence*

$$\mathcal{M}(S\times_K S,\mathbb{R})\xrightarrow{p_{1,*}-p_{2,*}}\mathcal{M}(S,\mathbb{R})\to\mathcal{M}(K,\mathbb{R})\to 0,$$

where $\mathcal{M}(K,\mathbb{R})$ is the space of Radon measures of K endowed with the weak topology.

Proof. Since $S\to K$ is a cover, we have a left exact sequence

$$0\to C(K,\mathbb{R})\to C(S.\mathbb{R})\to C(S\times_K S,\mathbb{R}). \tag{4.4}$$

But the proof of Theorem 4.1 shows that the dual of (4.4) is right exact. The corollary follows by noticing that $\mathcal{M}(K,\mathbb{R})=\underline{\mathrm{Hom}}_{\mathbb{R}}(C(S,\mathbb{R}),\mathbb{R})$ by Riesz representability theorem. ☐

5. $\mathcal{M}$-complete tensor product

We end by discussing a complete tensor product on $\mathcal{M}$-complete real vector spaces.

Proposition 5.1. *Let V and W be two $\mathcal{M}$-complete real vector spaces. Then there is an $\mathcal{M}$-complete real vector space $\mathcal{V}\otimes_{\pi}\mathcal{W}$ representing bilinear maps $V\times W\to L$ with values in quasi-separated $\mathcal{M}$-complete real vector spaces L.*

The functor $(V,W)\mapsto V\otimes_{\pi}W$ commutes with colimits in each variable and satisfies $\mathcal{M}(S_1,\mathbb{R})\otimes_{\pi}\mathcal{M}(S_2,\mathbb{R})=\mathcal{M}(S_1\times S_2,\mathbb{R})$ for S_i profinite sets.

Proof. It suffices to show that $\mathcal{M}(S_1\times S_2,\mathbb{R})$ represents bilinear maps from $\mathcal{M}(S_1,\mathbb{R})\times\mathcal{M}(S_2,\mathbb{R})$, the general case follows by writing Smith spaces as the cokernel of Radon measures of profinite sets, and writing an $\mathcal{M}$-complete real vector space as colimit of Smith spaces. Thus, we need to show that bilinear maps $\mathcal{M}(S_1,\mathbb{R})\times\mathcal{M}(S_2,\mathbb{R})\to L$ are the same as maps $S_1\times S_2\to L$. Any map $S_1\times S_2\to L$ already extends to a map $\mathcal{M}(S_1\times S_2,\mathbb{R})\to L$, and induces a bilinear map by taking the composition

$$\mathcal{M}(S_1,\mathbb{R})\times\mathcal{M}(S_2,\mathbb{R})\to\mathcal{M}(S_1\times S_2,\mathbb{R})\to L.$$

Thus, it suffices to see that any bilinear map $B:\mathcal{M}(S_1,\mathbb{R})\times\mathcal{M}(S_2,\mathbb{R})\to L$ that vanishes at $S_1\times S_2$ is already 0. This follows from the fact that the $\mathbb{R}$-span of S_i on $\mathcal{M}(S_i,\mathbb{R})$ is dense. ☐

The previous completed tensor product is naturally related with the projective and injective tensor products of Banach spaces.

Proposition 5.2. *Let V_1 and V_2 be two Banach spaces, then $V_1 \otimes_\pi V_2$ is the underlying Banach space of the projective tensor product of Banach spaces. Conversely, let W_i be the Smith dual of V_i, $W = V_1 \otimes_\pi V_2$ and V the Banach dual of W. Then there is a closed immersion from the injective tensor product of V_1 and V_2 into V, which is an isomorphism if one of the V_i satisfies the approximation property (i.e. if the identity map can be uniformly approximated by operators of finite rank).*

Proof. See [CS20, Propositions 4.10 and 4.12], note that the proof for the projective tensor product reduces to the same computation that we have made when identifying the solid tensor product of two Banach spaces over $\mathbb{Q}_p$. $\square$

References

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`, 2020.