Lie Groups: Fall, 2022
Problems for Lecture VI

November 29, 2022

Problem 1. Show that $\mathfrak{so}(4) \cong \mathfrak{sl}(3)$. Show $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$.

Problem 2. Show that $\mathfrak{so}(5) \cong \mathfrak{sp}(4)$.

Problem 3. Let $\mathfrak{g}$ be a Lie algebra. Suppose that $I \subset \mathfrak{g}$ is a nilpotent ideal and the quotient $\mathfrak{g}/I$ is a nilpotent. Show that $\mathfrak{g}$ is solvable. Show that $\mathfrak{g}$ is nilpotent if and only if for every $x \in \mathfrak{g}$ the restriction of $ad_{\mathfrak{g}}(x)$ to I is nilpotent.

Problem 4. Show that with the presentation fo $\mathfrak{sp}(2n)$ given in the lecture, show that the compact form of $\mathfrak{sp}(2n)$ is $\mathfrak{sp}(2n) \cap \mathfrak{su}(2n)$.

Problem 5 Let $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum of two commuting Lie algebras. Show that the radical, resp. nilradical, of the product is the product of the radicals, resp. nilradicals, of the factors.

Problem 6. Let (V, B, Φ) be a root system. We say that two roots α, β are *elementarily equivalent* if $\langle \alpha, \beta \rangle \neq 0$. We define the equivalence relation generated by elementary equivalence. For each equivalence class $\Phi_a \subset \Phi$ let V_a be the vector subspace of V generated by Φ_a and let $B_a = B|_{V_a}$. Show that the V_a are mutually orthogonal under B. Show that for each equivalence class a, the triple (V_a, B_a, Φ_a) is a root system and that this decomposes (V, B, Φ) is an orthogonal direct sum of root systems.

Problem 7. Suppose that the root system in Problem 6 is the root system of a semi-simple Lie algebra $\mathfrak{g}$. Show that the factors of the decomposition of the root system as given in Problem 6 are the root systems of the simple factors of $\mathfrak{g}$.

Problem 8. Show that for a semi-simple algebra $\mathfrak{g}$ we have $\mathcal{D}_m(\mathfrak{g}) = \mathfrak{g}$ for all $m \geq 1$.