# Problem Set 4 for Lie Groups: Fall 2022

October 25, 2022

**Problem 1.** Show that $Spin(2) \to SO(2)$ is the double covering of the circle over the circle. Show $Spin(3)$ is $SU(2)$. Show that $Spin(4) \cong SU(2) \times SU(2)$.

**Problem 2.** Let $Spin(1,3)$ be the Spin group associated with the real quadratic form of type $(1,3)$. Show that $Spin(1,3) \cong SL(4,\mathbb{C})$. The Poincaré group is defined as

$$Spin(1,3) \ltimes \mathbb{M}^{1,3}$$

where $\mathbb{M}^{1,3}$ is Minowski 4-dimensional space and with $Spin(1,3)$ acing via the 'rotation' group $SO(1,3)$ of isometries fo $\mathbb{M}^{1,3}$ fixing the origin. Much of physics revolves around unitary representations of the Poincaré group.

**Problem 3.** Let $k$ be a subfield of $K$. Suppose that $V$ is a $k$-vector space with a non-degnerate quadratic form $Q$. Then show that by extension of scalars there is an induced non-degenerate quadratic form $\hat{Q}$ on $V \otimes_k K$ and that there is a natural map $ClLV, Q) \otimes_k K \to CL(V \otimes_k K, \hat{Q})$.

**Problem 4.** Show that all non-degenerate quadratic forms on an $n$-dimensional complex vector space are isomorphic. We denote by $CL_{\mathbb{C}}(n)$ the (complex) Clifford algebra associated to a non-degenerate $n$-dimensional complex quadratic form. Show $CL_{\mathbb{C}}(n+2) \cong CL_{\mathbb{C}}(n) \otimes CL_{\mathbb{C}}(2)$. Give the classification of all finite dimensional complex Clifford algebras.

**Problem 5.** Let $k$ be a field of characteristic $\neq 2$. Show that a non-degenerate quadratic form on a finite dimensional $k$-vector space $V$ can be diagonalized; i.e., there is a basis $\{e_1, \ldots, e_k\}$ for $V$ such that under the bilinear form $B$ associated to $Q$ we have $B(e_i.e_j) = 0$ for all $i \neq j$. Show that the isomorphism classes of one-dimensional quadratic forms over $k$ are identified with $k^\times/(k^\times)^2$. What is the Clifford algebra associated with the one-dimensional form over $k$ determined by $d \in k^\times$.

**Problem 6.** Verify the claims in the last paragraph in Section 1.2.1 of Lecture IV.

**Problem 7.** Establish the statement about the existence of $e_2''$ in the last paragraph of Section 2.2.1 of Lecture IV.

**Problem 8.** We have identified $CL_{3,1}^0$ with $\mathbb{C}[2]$ and $CL_{3,1}$ with $\mathbb{H}[2]$. Describe the map $\mathbb{C}[2] \to \mathbb{H}[2]$ induced by these identifications and the natural inclusion $CL_{3,1}^0 \to CL_{3,1}$.

**Problem 9.** Same question as Problem 8 for the identifications $CL_{1,3} = \mathbb{R}[4]$ and $Cl_{1,3}^0 = \mathbb{C}[2]$.

**Problem 10.** All tensor products in this problem are over $\mathbb{R}$ and *isomorphism* means isomorphism of $\mathbb{R}$-algebras. Show that $\mathbb{R}[r] \otimes \mathbb{R}[s]$ is isomorphic to $\mathbb{R}[rs]$. Show that $K \otimes \mathbb{R}[s]$ is isomorphic to $K[s]$ for $K = \mathbb{C}, \mathbb{H}$. Show that $\mathbb{C} \otimes \mathbb{H}$ is isomorphic to $\mathbb{C}[2]$. Show that $\mathbb{H} \otimes \mathbb{H}$ is isomorphic to $\mathbb{R}[4]$.