Problem Set 1 for Lie Groups: Fall 2022

August 24, 2022

Problem 1. (a) Let R be a ring (with unit). The *units* in R, denoted R^* are the elements with a multiplicative inverse. Show that under the ring multiplication R^* forms a group. (b) Describe $\mathbb{Z}^*$, k^* for k a field. H^* for H the quaternions. (c) Let R be a commutative ring. We say that $r \in R$ is a *root of unity* if there is an integer $n > 0$ with $r^n = 1$. Show that the roots of unity in R form a subgroup of R^*.

Problem 2. Fix a field K and consider the polynomial ring with variables X_{ij} with $1 \leq i, j \leq n$. This is the ring of polynomial functions (with coefficients in K) on K^{n^2}. We identify the space $M(n \times n, K)$ of $n \times n$ matrices over K with the vector space K^{n^2} in such a way that the function $X_{i,j}$ assigns to each matrix its (i, j)-entry. For any r, we give K^r the *Zariski* topology where the closed subsets are exactly the loci in K^r where some given collection of polynomial vanishes. (These are called *subvarieties*.) Show that this is indeed a topology. Show that $GL(n, K) \subset M(n \times n, K)$ is open in the Zariski topology. Show that $SL(n, K)$, the matrices of determinant 1, is a closed subset in the Zariski topology. Show that matrix multiplication is an algebraic map, meaning that it pulls back polynomial functions on $M(n \times n, K)$ to polynomial functions on $M(n \times n, K) \times M(n \times n, K)$. Show that this map is cotninuous in the Zariski topology. Show that $GL(n, K)$ and $SL(n, K)$ are groups under matrix multipliciation.

Problem 3. A *linear algebraic group over* K is subvariety V of $M(n \times n, K)$ that is closed under multiplication and taking inverses. Show that any linear algebraic group V in $M(n \times n, K)$ is a subvariety contained in $GL(n, K)$. Show that multiplication and inverse are given by rational functions where the demoninator is a power of the determinant and hence does not vanish on V. Show that $GL(n, K)$ is a linear algebraic group in the sense that there is a linear algebraic group V in $M(n' \times n', K)$ for some n' and an isomorphism $GL(n, K) \to V$ of K-algebraic varieties that commutes with multiplication and inverses. Show that the polynomial functions on $GL(n,K)$ are polynomials in the variables X_{ij} and $\det^{-1}$. Show that any linear algebraic group over $\mathbb{R}$, resp. $\mathbb{C}$, is a Lie group, resp. a complex Lie Group.

Problem 4. Show that the additive group $(K,+)$ and the multiplicative group $(K^*,\cdot)$ are linear algebraic groups. Show that $(\mathbb{R},+)$ and $(\mathbb{R}^*,\cdot)$ are Lie Groups and $(\mathbb{C},+)$ and $(\mathbb{C}^*,\cdot)$ are complex Lie Groups.

Problem 5. Let $\Lambda \subset \mathbb{C}$ be a lattice spanned by two vectors linearly independent over $\mathbb{R}$. Show that the quotient $\mathbb{C}/\Lambda$ is a compact complex manifold. Show that addition on $\mathbb{C}$ descends to define a (complex) Lie group structure on $\mathbb{C}/\Lambda$. Show that this complex Lie group is not isomorphic (as a complex Lie group) to any linear algebraic group defined over $\mathbb{C}$.

Problem 6. Let Q be a positive definite quadratic form on a n-dimensional real vector space. Show that the orthogonal group of Q is a Lie group by showing that it is a smooth submanifold of $GL(n,\mathbb{R})$. [Hint: Show that there is no loss of generality in taking Q to be the standard Euclidean quadratic form. Then show a matrix $A \in M(n\times n,\mathbb{R})$ is in the orthogonal group if and only if its columns form an orthonormal basis of $\mathbb{R}^n$. Then use the implicit function theorem to show establish the result.]

Problem 7. For Q the standard Euclidean quadratic form on $\mathbb{R}^n$ determine the Lie algebra of its orthogonal group $O(n)$.

Problem 8. Show that the group of unitary $n\times n$-matrices, i.e., $A \in GL(n,\mathbb{C})$ satisfying $\overline{A}^{tr} = A^{-1}$ is a real Lie subgroup of $GL(n,\mathbb{C})$. Show that in general it is not a complex Lie subgroup. What is its Lie algebra in $M(n\times n,\mathbb{C})$. Same questions for the special unitary subgroup – the group of unitary matrices of determinant 1.

Problem 9. Let A be a non-degenerate skew symmetric pairing on $\mathbb{R}^{2n}$. Define $Symp(2n,\mathbb{R})$ as the set of elements in $g \in GL(2n,\mathbb{R})$ that preserve A in the sense that $A(x,y) = A(gx,gy)$ for all $x,y \in \mathbb{R}^{2n}$. Show that $Symp(2n,\mathbb{R})$ is a sub-Lie Group of $GL(n,\mathbb{R})$. Compute its Lie algebra in $M(n\times n,\mathbb{R})$.

Problem 10. Show that the Lie algebra of $SL(n,\mathbb{R})$ is the subspace of $M(n\times n,\mathbb{R})$ of trace-zero matrices.

Problem 11. Let H be the group of upper triangular 3×3 real matrices with 1's down the diagonal. This is the *Heisenberg group*. Show that this is a real Lie group, indeed a linear algebraic group over $\mathbb{R}$. Show that the map

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a,b) \in \mathbb{R} \times \mathbb{R}$$

is a surjective homomorphism of Lie groups with kernel equal to the center of H. Compute the Lie algebra $\mathfrak{h}$, the *Heisenberg Lie algebra.* Show that $\mathfrak{h}$ is a *nilpotent* Lie algebra in the sense that there is $k \geq 1$ so that all brackets of length $\geq k$ in $\mathfrak{h}$ vanish. Show that the exponential map $\mathfrak{h} \to M(3 \times 3, \mathbb{R})$ defined by $A \mapsto \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is a polynomial function and gives a diffeomorphism between $\mathfrak{h}$ and H. Show that there is a version of this over $\mathbb{Q}$: the exponential series on $\mathfrak{h}$ is a polynomial with $\mathbb{Q}$ coefficients and defines an isomorphism of algebraic varities over $\mathbb{Q}$ from the rational form of $\mathfrak{h}$ to the linear algebraic group $H_{\mathbb{Q}} \subset M(3 \times 3, \mathbb{Q})$ given by the equations $X_{ij} = 0$ for all $i > j$ and $X_{ii} = 1$.

Problem 12. Prove Lemma 2.1 and show in addition that the kernel of $\pi\colon \widetilde{G} \to G$ is contained in the center of $\widetilde{G}$.