Lie Groups: Fall, 2022
Lecture V
Representations of $\mathfrak{sl}(2,\mathbb{C})$ and of Compact Lie Groups

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1 Representations of $\mathbb{C}^*$

Let us begin with a well-known theorem.

Proposition 1.1. *Let $\rho\colon \mathbb{C}^* \times V \to V$ be a finite dimensional complex linear representation. Then there is a basis $\{e_1, \ldots, e_k\}$ for V and integers $n_1, \ldots, n_k$ such that the action of $z \in \mathbb{C}^*$ sends e_i to $z^{n_i}e_i$. In other words in some basis the action is diagonal with integral powers of z down the diagonal.*

Proof. We choose coordinates w for $\mathbb{C}$ and z for $\mathbb{C}^*$ so that the exponential map is given by $z = \exp(2\pi i w)$.

Let $A \in \mathbb{C}[n]$ be the image of $1 \in \mathbb{C}$ under the differential of ρ. Then the action $\rho(z)$ is given by the matrix $\exp(2\pi i w A)$. The Jordan canonical form for A says that in some basis we can write $A = D + N$ where D is a diagonal matrix and N nilpotent, say $N^{k+1} = 0$, and $[D, N] = 0$. Since D and N commute, we have

$$\exp(2\pi i w A) = \exp(2\pi i w D)\exp(2\pi i w N).$$

Since N is nilpotent, $\exp(2\pi i w N)$ is a polynomial function

$$p(2\pi i w N) = \sum_{j=0}^{k} \frac{(2\pi i w N)^j}{j!}.$$

The first exponential is diagonal and the second is nilpotent matrix commuting with the diagonal term. It follows that the eigenvalues of $\exp(2\pi i w A)$

are the exponential of $2\pi i$ times the diagonal entries of D. Since the former are periodic under $w \mapsto w+1$, the latter are integers. Thus, $\exp(2\pi i w D)$ is a diagonal matrix with diagonal entries of the form $\exp(2\pi i\lambda + jw) = z^{\lambda_j}$ for integers λ_j.

Since $\exp(2\pi i w D)$ is periodic under $w \mapsto w+1$ as is $\exp(2\pi i w A)$, it follows that $p(2\pi i w N)$ is also periodic under $w \mapsto w+1$. Since only constant polynomials of w are periodic under $w \mapsto w+1$, this implies that the coefficient of each positive powers of w in $p(w)$ is identically zero. In particular, the coefficient of the linear term of w in $p(w)$, which is $2\pi i N$, is 0, and hence $A = D$ and A is thus diagonalizable with diagonal entries z^{λ_j} as asserted. $\square$

2 The Lie Algebra of $\mathfrak{s}l(2,\mathbb{C})$

$\mathfrak{s}l(2,\mathbb{C})$ is the Lie sub algebra of $\mathbb{C}[2]$ of matrices of trace zero. Generators for this algebra are

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The first matrix exponentiate a $\mathbb{C}^*$ in $SL(2,\mathbb{C})$. The relations in the Lie algebra are $[h, e_+] = 2e_+$, $[h, e_-] = -2e_-$, and $[e_+, e_-] = h$.

Definition 2.1. An *ideal* I in a Lie algebra L is a linear subspace I with the property that $[I, L] \subset I$. An ideal is *non-trivial* if it is neither all of L nor 0.

Claim 2.2. *If $\rho\colon L \to L'$ is a homomorphism of Lie algebras then the kernel of ρ is an ideal in L.*

The proof is left as an exercise.

Lemma 2.3. *$\mathfrak{s}l(2,\mathbb{C})$ has no non-trivial ideal.*

Proof. Let I be a non-zero ideal of $\mathfrak{s}l(2,\mathbb{C})$ and let $X \neq 0$ be an element of I. Then,$[e_-, [e_-, [e_-, X]]] = 0$. Let $0 \leq k < 3$ be the largest integer such that k-fold bracketing X by e_- is non-zero. Then this bracket, which is contained in I, is a non-zero multiple of e_-. This shows $e_- \in I$. Bracketing by e_+ we see that $h \in I$, and bracketing once more by e_+ we see that $e_+ \in I$. This proves that the ideal is all of $\mathfrak{s}l(2,\mathbb{C})$. $\square$

Suppose that we have a finite dimensional complex representation of $\mathfrak{sl}(2, C)$ on a vector space V. Since $\mathfrak{sl}(2, \mathbb{C})$ has no non-trivial ideals it follows that either this map is trivial or it is an embedding of $\mathfrak{sl}(2, \mathbb{C})$ into $\mathrm{End}(V)$. We assume that the representation is non-trivial. Then it generates a subgroup $G \subset \mathrm{Aut}(V) \cong GL(n, \mathbb{C})$. This group has the same Lie algebra as $SL(2, \mathbb{C})$ and thus is a finite quotient by a central subgroup. (The center of $SL(2, \mathbb{C})$ consists of the diagonal matrices with values ± 1 down the diagonal, and hence is a group of order 2.) In any case, the Lie algebra representation integrates to a non-trivial representation of $SL(2, \mathbb{C})$ (possibly with kernel $\{\pm 1\}$). As such we see that we can choose a basis for V in which the infinitessimal generator h is diagonal with integer entries along the diagonal.

Claim 2.4. *For $v \in V$, if $h(v) = \lambda v$, then $h(e_-(v)) = (\lambda - 2)e_-(v)$ and $h(e_+(v)) = (\lambda + 2)e_+(v)$.*

Proof. We apply the equation $[h, e_+] = 2e_+$ to v and conclude that $h(e_+(v)) - e_+(h(v)) = h(e_+(v)) - e_+(\lambda v) = 2e_+(v)$, giving $h(e_+(v)) = (\lambda + 2)e_+(v)$. The argument for e_- is symmetric. □

Now let λ be the maximal eigenvalue for the action of h on V and let $v \neq 0$ be a vector in V with $h(v) = \lambda(v)$. We see that $e_+(v) = 0$. For every $j \geq 0$ we define $v_j = (e_-)^j(v)$. Then $h(v_j) = (\lambda - 2j)(v_j)$.

Claim 2.5. *For any $j \geq 0$, if $v_j \neq 0$, then $e_+(v_{j+1}) = \sum_{i=0}^{j-1}(\lambda - 2i)v_j$.*

Proof. If $v_j \neq 0$, then $v_i \neq 0$ for all $0 \leq i \leq j$. We establish the formula in the claim by induction on i. It holds for $i = 0$ since

$$\lambda v_0 = [e_+, e_-](v_0) = e_+ e_-(v_0) = e_+(v_1).$$

Suppose that it holds for v_{j-1}. This means that $e_- e_+(v_j) = \sum_{i=0}^{j-1}(\lambda - 2i)v_j$ Thus,

$$(\lambda - 2j)v_j = (e_+ e_- - e_- e_+)(v_j) = e_+(v_{j+1}) - \sum_{i=0}^{j-1}(\lambda - 2i).$$

The result follow by induction. □

Corollary 2.6. *For any $t \neq \lambda$, if $v_t \neq 0$, then $v_{t+1} \neq 0$.*

Proof. As long as $t \neq \lambda$, the numerical coefficient is nonzero, and hence $v_{t+1} \neq 0$. □

Corollary 2.7. *λ is a non-negative integer and the dimension of the $\mathfrak{sl}(2,\mathbb{C})$ representation generated by v is $\lambda+1$.*

Proof. Since V is finite dimensional there can be only finitely many v_t that are non-zero. Hence, it must be the case that $v_{\lambda+1}=0$, for otherwise v_t would be non-zero for all t. On the other hand since $v_0 \neq 0$, $v_t \neq 0$ for all $0 \leq t \leq \lambda$. Thus, v belongs to a unique subrepresentation of V of dimension $\lambda+1$. $\square$

Indeed, the above argument does not need the eigenvalue of v to be maximal, all that is required is that $e_+(v)=0$. Since $he_+ - e_+h = 2e_+$, we see that h leaves invariant $\ker(e_+)$.

Thus, we write kernel(e_+) as a direct sum $\oplus_i L_i$ where each L_i is one-dimensional and contained in eigenspace of h with eigenvalue $\lambda_i \in \mathbb{Z}$. Then each L_i generates an $\mathfrak{sl}(2,\mathbb{C})$-submodule V_i of V of dimension λ_i+1.

Claim 2.8. *The inclusions $V_i \subset V$ determine an isomorphism of $\mathfrak{sl}(2,\mathbb{C})$ modules $\oplus_i V_i \to V$.*

Proof. Since each V_i is a subrepresentation of V, we have the sum of the inclusion maps $\rho\colon \oplus_i V_i \to V$ which is an $\mathfrak{sl}(2,\mathbb{C})$ homomorphism. By construction it induces an isomorphism on $\ker(e_+)$. As we have seen, every non-trivial finite dimensional $\mathfrak{sl}(2,\mathbb{C})$ representation contains a non-trivial vector in $\ker(e_+)$. Thus, if $\ker(\rho) \neq 0$, then there would be a non-trivial element in $\ker(e_+)$ which maps trivially under ρ, which is not possible. This contradiction proves the map $\oplus_i V_i \to V$ is a monomorphsm.

Now let $W \subset V$ be the image of $\oplus V_i$. Assume that there is a non-trivial cokernel. Then there is an elementary subrepresentation generated by an element in V/W in the kernel of e_+. In particular, there is an element $v \in V$, not in W, that is in non-negative eigenspace for h and has $e_+(v) \in W$. The element $e_+(v)$ is in a positive h-eigenspace of W. Direct computation shows that the elementary summands of W split as vector spaces as $\mathrm{Im}(e_+) \oplus \ker(e_-)$ which means in particular that $e_+(W)$ contains all the positive h-eigenspaces on W. Thus, $e_+(v)=e_+(w)$ for some $w \in W$. Hence, $e_+(v-w)=0$ and $v-w \notin W$. This is a contradiction since the map $W \to V$ induces an isomorphism of the kernels of e_+. $\square$

Definition 2.9. A finite dimensional representation V of a Lie algebra L is said to be *completely reducible* if it is isomorphic as a L-module to a direct sum of irreducible L-modules, i.e. L-modules that admit no non-trivial submodules.

Theorem 2.10. *Every finite dimension $\mathfrak{sl}(2,\mathbb{C})$-module is completely reducible. Up to isomorphism there is exactly one irreducible $\mathfrak{sl}(2,\mathbb{C})$ module for dimension k, for every $k \geq 1$. It is generated by a vector v with $h(v) = (k-1)v$ and $e_+(v) = 0$. A $\mathbb{C}$-basis for the module is $\{v, e_-(v), \ldots (, e_-)^{k-1}v\}$.*

What are these representations? By the general theory, we know that they all come from representations of $SL(2,\mathbb{C})$. The one-dimensional representation is the trivial representation on $\mathbb{C}$. The two-dimensional representation is the defining representation $SL(2,\mathbb{C}) \subset GL(2,\mathbb{C})$. The higher dimensional representations are the symmetric powers of the defining representation. The symmetric k^{th} power has dimension $(k+1)$, with a basis $e_1^a e_2^b$ with $a+b=k$. (Here, e_1, e_2 is a basis for the two-dimensional representation.) We can also view these representations as the induced representation on homogenous polynomial functions of degree k on the two-dimensional representation.

The even dimensional irreducible representations are faithful representations of $SL(2,\mathbb{C})$. The reason is that all the eigenvalues of h are odd integers and thus the resulting action of the $\mathbb{C}^*$ generated by h is a diagonal matrix with odd powers of z down the diagonal. This map is a faithful action of $\mathbb{C}^*$. Since the only possible kernel for the action of $SL(2,\mathbb{C})$ is the central subgroup ± 1 that lies in the center, any action of $SL(2,\mathbb{C})$ whose restriction to the diagonal $\mathbb{C}^*$ is faithful is a faithful action of $SL(2,\mathbb{C})$.

The odd dimensional irreducible representations have h eigenvalues that are even integers, and hence the resulting action of $\mathbb{C}^*$ is diagonal with even powers of z along the diagonal. Any such action is trivial on the element -1, and hence the representation factors through $SL(2,\mathbb{C})/\{\pm 1\}$.

3 $\mathfrak{sl}(2,\mathbb{R})$

Any complex representation of $\mathfrak{sl}(2,\mathbb{R})$ extends to a representation of $\mathfrak{sl}(2,\mathbb{C})$. As such, $h \in \mathfrak{sl}(2,\mathbb{R})$ is diagonalizable with integer eigenvalues. The argument given in for $\mathfrak{sl}(2,\mathbb{C})$ now applies word-for-word to show that all finite dimensional complex representation of $\mathfrak{sl}(2,\mathbb{R})$ is completely irreducible and up to isomorphism the irreducible representations are the symmetric powers of the defining 2-dimensional complex representation.

Now let us consider finite dimensional, real representations of $\mathfrak{sl}(2,\mathbb{R})$. Fix one V. We first pass to the complex representation V_C of $\mathfrak{sl}(2,\mathbb{R})$. In this complex representation, as we have just seen, h is diagonalizable with integer eigenvalues. Since the eigenvalues are all real, this implies that there is a (real) basis of V in which h is diagonal. Once we have this the arguments in the previous cases apply to show that the representation is completely reducible and the irreducible representations are the symmetric powers of the defining real representation. Thus, all these representations come from the symmetric powers of the defining 2-dimensional real representation of $SL(2,\mathbb{R})$. As before the even dimensional irreducible representations of $SL(2,\mathbb{R})$ are faithful and the odd dimensional representations factor through $SL(2,\mathbb{R})/\{\pm 1\}$.

4 $\mathfrak{so}(3)$

The Lie algebra $\mathfrak{so}(3)$ is the 3-dimensional real Lie algebra of skew symmetric 3×3 real matrices. It has a basis X, Y, Z and bracket relations $[X,Y] = Z$, $[Y,Z] = X$, $[Z,X] = Y$. It is the Lie algebra of the compact group $SO(3)$, and the universal covering of $SO(3)$ is $SU(2)$, the 2×2 unitary matrices of determinant 1. Thus any representation of $so(3)$ determines a representation of $SU(2) = Spin(3)$.

Claim 4.1. *The complexification of $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. While we can give a direct argument, using the fact that $\mathfrak{so}(3)$ is the Lie algebra of $SU(2)$ and we have $SU(2)\subset SL(2,\mathbb{C})$. These together induce an isomorphism between the complexification of the Lie algebra of $SU(2)$ with $\mathfrak{sl}(2,\mathbb{C})$. $\square$

It follows that the complex representations of $\mathfrak{so}(3)$, which are the same as the complex representations of $\mathfrak{so}(3)\otimes\mathbb{C}$. (It follows that that the complex representations of $SU(2)$ and $SO(3)$ are completely reducible.) The complex representations of these Lie algebras and Lie groups are completely reducible. There is one irreducible complex representation of $SU(2)$ in each dimension. Each of these is a complex representation of a map of complex Lie groups $SU(2)\to GL(N,\mathbb{C})$. The even dimensional irreducible complex representations are faithful and the odd dimensional ones factor to give irreducible complex representations of $SO(3)$. This gives a complete description of all finite dimensional representations of $SU(2)$ and $SO(3)$.

4.1 Real Representations

Let's begin with a simple example that shows that not every complex representation of $\mathfrak{so}(3)$ comes from a real representation.

Claim 4.2. *There is no irreducible 2-dimensional real representation of $\mathfrak{so}(3)$.*

Proof. Since $\mathfrak{so}(3)$ is the Lie algebra of $SO(3)$ and hence also the Lie algebra of its double cover S^3 with group multiplication as the quaternions of unit length, any real representation of $\mathfrak{so}(3)$ comes from a representation of the Lie group S^3. Let V be a finite dimensional real representation of $\mathfrak{so}(3)$ and hence of S^3. Fix any positive definite quadratic from Q on V and consider

$$\overline{Q} = \int_{S^3} g^* Q dg$$

where dg is the usual volume element on S^3 thought of as the unit sphere in $\mathbb{R}^4$. Then $\overline{Q}$ is a positive definite form on V invariant under the action of S^3. Thus, S^3 is represented as the isometries of $(V, \overline{Q})$, and the differential of the representation is a homomorphism of Lie algebras $\mathfrak{so}(3) \to \mathfrak{so}(n)$ where $n = \dim(V)$.

If the dimension of V is 2, then $\mathfrak{so}(2)$ is an abelian Lie algebra and hence any representation of $\mathfrak{so}(3) \to \mathfrak{so}(2)$ induces a homomorphism $S^3 \to S^1$. The only such homomorphism is trivial. Thus, the only two dimensional real representation of $\mathfrak{so}(3)$ is trivial and hence not irreducible. □

In fact, the classification of real representations of $\mathfrak{so}(3)$ is known. All the odd dimensional irreducible complex representation of $\mathfrak{so}(3)$ come from real representations of $SO(3)$ built out of the defining 3-dimensional representation of $SO(3)$. The only even dimensional irreducible complex representations of $\mathfrak{so}(3)$ that come from real representations are those that are induced real representations underlying quaternion representations. These of course all have dimension congruent to zero modulo 4. As we have seen, there is no irreducible real 2-dimensional representation; more generally there is no irreducible real representation of dimension $4k + 2$ for any $k \geq 0$. We shall not prove this.

5 Complete Reducibility of Representations of Compact Lie Groups

The arguments in the last section showing finite dimensional linear representations of $SU(2)$ are completely reducible has a vast generalization.

Theorem 5.1. *Let G be a compact group and $\rho\colon G \times V \to V$ a finite dimensional complex linear representation. This representation is completely reducible.*

Proof.

Lemma 5.2. *The compact Lie group G has a Riemannian metric that is invariant under left translation.*

Proof. Fix a positive definite quadratic form $Q_e \colon \mathfrak{g} \to \mathbb{R}$ and define a smooth section of the bundle of quadratic forms on the tangent bundle by setting $Q_g \colon T_gG \to \mathbb{R}$ equal to the composition

$$T_gG \xrightarrow{g^{-1}\cdot} \mathfrak{g} \xrightarrow{Q} \mathbb{R}.$$

Let $B \colon TG \times_G TG \to \mathbb{R}$ be the associated bilinear form. Then B is a smoothly varying, positive definite bilinear form on the tangent spaces to G. From the definition one sees that B is invariant under left multiplication in the sense that for tangent vectors $\tau_1, \tau_2 \in T_hG$

$$B(\tau_1, \tau_2) = B(g \cdot \tau_1, g \cdot \tau_2).$$

That is to say, B a Riemannian metric on G invariant under left multiplication. $\square$

Lemma 5.3. *G has a Borel measure invariant under left-multiplication.*

Proof. Choose an orientation for $\mathfrak{g}$ and hence an orientation $\mathcal{O}$ for G. (This orientation is left-invariant.) Denote by n the dimension of G. Associated with the Riemannian metric B and the orientation is a differential n-form $\omega_{B,\mathcal{O}}$ whose integral over n-tangent vectors $\{\tau_1, \cdots, \tau_n\}$ at a point $h \in G$ is the signed volume of the parallelopiped spanned by these vectors. Said another way, fix an orthonormal basis $\{e_1, \ldots, e_n\}$ giving the orientation of T_hG Then $\tau_1 \wedge \cdots \wedge \tau_n = V \cdot e_1 \wedge \cdots \wedge e_n$ for some $V \in \mathbb{R}$. Then

$$\omega_{B,\mathcal{O}}(\tau_1, \ldots, \tau_n) = V.$$

Now we define μ to be the Borel measure as follows. If $U \subset G$ is an open set, then

$$\mu(U) = \int_U \omega_{B,\mathcal{O}}$$

where U is oriented as an open submanifold of G. This measure is left-invariant in the sense that $\mu(T) = \mu(g \cdot T)$ for all measurable sets T. $\square$

Lemma 5.4. *Now let $\rho \colon G \times V \to V$ be a finite dimensional complex representation. Then V has a positive definite Hermitian inner product invariant under the left action of G.*

Proof. Give V a positive definite hermitian inner product, denoted $\langle v_1, v_2\rangle_0$. We form a new pairing $\langle\cdot,\cdot\rangle$ on V by defining

$$\langle v_1, v_2\rangle = \int_G \langle gv_1, gv_2\rangle_0 d\mu.$$

This is a real bilnear form that is complex linear in the first variable and complex anti-linear in the second with $\langle v, v\rangle > 0$ for any non-zero vector v. That is to say, $\langle\cdot,\cdot\rangle$ is a positive definite Hermitian inner product on V. Clearly since μ is left-invariant this Hermitian inner product on V is left-invariant under the G-action. $\square$

Now we prove the complete reducibility by induction on the dimension of the representation. It is clear for 1-dimensional representations since there are no non-trivial linear subspaces. Suppose that we know the result for representations of dimension $< n$ and V is an n-dimensional representation.

If V is irreducible, then there is nothing to prove. So suppose that V is reducible with a non-trivial G-invariant subspace $W \subset V$. We fix a G-invariant positive definite Hermitian inner product on V. Let $W^\perp$ be its orthogonal complement under the positive definite Hermitian inner product. Then $V = W \oplus W^\perp$ and each of W and $W^\perp$ are non-zero (since W is non-trivial.) Because the Hermitian inner product is G-invariant, if $v \in W^\perp$, i.e. $\langle w, v\rangle = 0$ for all $w \in W$, then for any $g \in G$ $\langle gw, gv\rangle = 0$ for all $w \in W$. But the left action of g is a linear isomorphism $W \to W$, so that it follows that $\langle w, gv\rangle = 0$ for all $w \in W$. That is to say, $gv \in W^\perp$. This shows that $W^\perp$ is G-invariant.

Thus, we have a decomposition of G representations $V = W \oplus W^\perp$. Since W and $W^\perp$ are both non-zero, each has smaller dimension than V. Hence by induction, each of W and $W^\perp$ are completely reducible and hence so is V. $\square$

Appendix: Borel measures

A *σ-algebra* in X is a collection of subsets Σ of X that (i) $X \in \Sigma$, (ii) if $A \in \Sigma$ then the complement A^c of A is also an element of Σ, and (iii) if $A_n \in \Sigma$ for $1 \le n < \infty$, then $A = \cup_n A_n$ is an element of Σ.

A *Borel σ-algebra* of a topological space is the σ-algebra generated by the open subsets of X. A *Borel measure* μ on X is a function from Borel measurable subsets of X to $[0, \infty]$ that satisfies additivity for countable disjoint unions of Borel measurable sets. We also require that the measure of any compact set is finite. A Borel measure μ for a locally compact metric space is inner and outer regular in the following sense. For any Borel measurable set A and any $\epsilon > 0$ there is a closed set $F \subset A$ and an open subset U with $A \subset U$ with $\mu(U \setminus A) < \epsilon$. In fact, this is a characterization of Borel measurable sets: ones that can be approximated from without by an open set U and within by a closed set F so that the difference in measure between U and F is at most ϵ. A Borel measure on a locally compact metric space is determined by its values on open subsets. Indeed, the measure of any Borel measurable A is the (decreasing) limit of Borel measure of open sets that are better and better outer approximations.

If f is any non-negative continuous function on $\mathbb{R}^n$, then we define a Borel measure by defining

$$\mu_f(U) = \int_{\mathbb{R}^n} f(z^1, \ldots, x^n) dx^1 \cdots dx^n.$$

Standard theorems in Lebesgue integration show that this is a Borel measure and finite on compact subsets We can restrict this to a Borel measure on any open subset $U \subset \mathbb{R}^n$ by replacing f by $f\chi_U$, where χ_U is the characteristic function of U. This function is not continuous but is Borel measurable, again by standard results in Lebesgue measure theory.

These results extend to smooth manifolds. Let M be a smooth, oriented manifold of dimension n and ω is nowhere 0 differential n form M compatible with the orientation. Then for any open subset $U \subset M$, we define

$$\mu(U) = \int_M \chi(U)\omega,$$

where U has the orientation induced from M. Straight forward extensions of the standard results in Lebesgue integration show that μ is a Borel measure, finite on every compact set.