Lie Groups: Fall, 2022
Lecture III
The Exponential Map, Local Lie Groups, and the Statement of the Baker-Canmpbell-Hausdorff Formula

September 26, 2022

In this lecture we shall make a deeper study of the relationship of a Lie group and its Lie algebra. We shall outline a proof of the fact that two simply connected Lie groups with isomorphic Lie algebras are themselves isomorphic. More generally, we show that if G is a Lie group and L is a sub Lie algebra of G then there is a simply connected Lie group H with Lie algebra isomorphic to L and a morphism of Lie groups $H \to G$ whose differential at the identity identifies the Lie algebra of H with L.

1 The Exponential Mapping.

We have shown how to pass from a Lie group to its Lie algebra by differentiating at the identity element (twice) the conjugation map of G on itself. The basic constructing passing from a Lie algebra to 'its' Lie group is the exponential mapping. This mapping identifies a neighborhood of the origin in $\mathfrak{g}$ with a neighborhood of the identity in G.

1.1 The case of $GL_n(\mathbb{R})$

Since $GL(n,\mathbb{R})$ is an open subset of $M(n\times n,\mathbb{R})$, any $A \in M(n\times n,\mathbb{R})$ determines a tangent vector to $GL(n,\mathbb{R})$ at the identity element. This identifies $M(n\times n,\mathbb{R})$ with $\mathfrak{gl}(n,\mathbb{R})$. The power series

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

converges absolutely for all $t \in \mathbb{R}$ and hence defines a smooth curve $\gamma_A(t)$ in $M(n \times n, \mathbb{R})$. By construction it satisfies $\gamma_A(0) = \mathrm{Id}$ and $\gamma'_A(0) = A$. The usual power series manipulations show that for all $t_1, t_2 \in \mathbb{R}$ we have $\gamma_A(t_1)\gamma_A(t_2) = \gamma_A(t_1 + t_2)$. Since $\gamma_A(t) \in GL(n, \mathbb{R})$ for all $|t|$ sufficiently small, it follows that, for all $t \in \mathbb{R}$ the matrix $\gamma_A(t)$ is contained in $GL(n, \mathbb{R})$, and furthermore, γ_A is a homomorphism of Lie groups $(\mathbb{R}, +) \to GL(n, \mathbb{R})$. We define the exponential map

$$\exp\colon \mathcal{M}(n \times n, \mathbb{R}) \to GL(n, \mathbb{R})$$

by

$$\exp(A) = \gamma_A(1) = e^A.$$

This is a smooth map from $M(n \times n, \mathbb{R}) \to GL(n, \mathbb{R})$ whose differential at the origin is the identity. By the implicit function theorem there is a neighborhood U of $0 \in M(n \times n, \mathbb{R})$ that maps diffeomorphically onto an open subset $\exp(U)$ of the identity in $GL(n, \mathbb{R})$. The inverse map is the logarithm $\log\colon \exp(U) \to U$.

In the case of $GL(n, \mathbb{C})$ the exponential map (given by the same power series) Associates to each $A \in M(n \times n, \mathbb{C})$ a homomorphism of Lie groups $\gamma_A\colon (\mathbb{C}, +) \to GL(n, \mathbb{C})$ with $\gamma'_A(0)\colon \mathbb{C} \to M(n \times n, \mathbb{C})$ the complex linear map sending $1 \in \mathbb{C}$ to A. We define a holomorphic map $\exp\colon \mathfrak{g}l(n, \mathbb{C}) \to GL(n, \mathbb{C})$ to send A to e^A. Analogously, the differential of this map at $0 \in \mathfrak{g}l(n, \mathbb{C})$ is the identity so that it is a local holomorphic isomorphism from some neighborhood of 0 in $\mathfrak{g}l(n, \mathbb{C})$ to an open neighborhood of the identity in $GL(n, \mathbb{C})$.

1.2 The Exponential Map for a General Lie Group

Theorem 1.1. *Let G be a Lie group. Then for every $A \in \mathfrak{g}$ there is a unique morphsim of Lie groups $\gamma_A\colon (\mathbb{R}, +) \to G$ with the property that $\gamma'_A(0) = A$.*

Proof. Fix $A \in \mathfrak{g}$. Let χ_A be the left-invariant vector field whose value at $g \in G$ is $g \cdot A$. By the existence and uniqueness results for ODEs, for some $\epsilon > 0$, there is a unique integral curve $\gamma_A\colon (-\epsilon, \epsilon) \to G$ for this vector field whose value at 0 is e. .

Claim 1.2. *The maximal interval of definition for the integral curve γ_A is the entire real line.*

Proof. By the existence theorem for solutions to ODEs, there is $\epsilon > 0$ such that γ_A is defined on $(-\epsilon, \epsilon)$. By uniqueness of solutions to ODEs, if I and J

are intervals of definition for an integral curve of χ_A, both containing 0, then the integral curves defined on these two intervals agree on the intersection of the intervals and hence the two curves define an integral curve on $I \cup J$. From this it is easy to see that there is a maximal interval of definition for the integral curve γ_A. We must show that this is $\mathbb{R}$.

Let $I \subset \mathbb{R}$ be the maximal interval of definition for γ_A and suppose that I is bounded above. Fix t_0 within $\epsilon/2$ of the least upper bound of I. Consider the curve $\mu(t_0+t) = \gamma_A(t_0)\gamma_A(t)$ for $t \in (-\epsilon, \epsilon)$. Then $\mu'(t_0+t) = \gamma_A(t_0)\gamma_A'(t) = \gamma_A(t_0)\gamma_A(t) \cdot A$. This shows that μ is an integral curve for χ_A. Since it and γ_A agree at t_0, they agree on their common domain of definition. This is a contradiction since it allows us to extend the domain of definition beyond the least upper bound of I and I was assumed to be the maximal interval of definition for the integral curve. Consequently, the interval I has no upper bound. Symmetrically, I has no lower bound. The only interval with no upper and no lower bound is $\mathbb{R}$. □

Analogously to the case of $GL(n, \mathbb{R})$, the differential equation shows that γ_A is a Lie group homomorphism from $(\mathbb{R}, +)$ to G.

Claim 1.3. *Suppose that* $\gamma\colon (\mathbb{R}, +) \to G$ *is a homomorphism of Lie groups and suppose that* $\gamma'(0) = A$. *Then* $\gamma(t) = \gamma_A(t)$ *for all* $t \in \mathbb{R}$.

Proof. Since γ is a homomorphism, it follows that $\gamma'(t) = \gamma(t)\gamma'(0)$, and thus γ is an integral curve for χ_A whose value at $t = 0$ is the identity. There is only one such integral curve and it is γ_A. □

This completes the proof of Theorem 1.1 □

Definition 1.4. We define *the exponential map*, $\exp_G\colon \mathfrak{g} \to G$ by sending $A \in \mathfrak{g}$ to $\gamma_A(1)$ where γ_A is the one-parameter subgroup whose tangent vector at the identity is A.

The following is clear from the definition.

Proposition 1.5. *The exponential mapping is a smooth map whose differential at* $0 \in \mathfrak{g}$ *is the identity. Hence, there is a neighborhood* $U \subset \mathfrak{g}$ *of* 0 *such that* $\exp_G$ *is a diffeomorphism from* U *to an open neighborhood* $\exp_G(U)$ *of the identity in* G. *We denote the inverse by* $\log\colon \exp(U) \to U$.

Corollary 1.6. *If* $H \subset G$ *is a Lie subgroup with Lie algebra* $\mathfrak{h} \subset \mathfrak{g}$*, then* $\exp_G|_{\mathfrak{h}} = \exp_H$. *Any particular, any one-parameter subgroup tangent to* H *at the origin is contained in* H.

Proof. For any $A \in \mathfrak{h}$ the left-invariant vector field gA is tangent to H. Hence the integral curve γ_A that passes through e at $t = 0$ lies in H. □

2 Local Lie Groups

There is an intermediate category between the categories of Lie groups and Lie algebras. It is the category of local Lie groups – germs of Lie groups at the identity.

Definition 2.1. A *local Lie Group* consists of:

(i) a smooth manifold U

(ii) an element $e \in U$,

(iii) a diffeomorphism $\theta \colon U \to U$ fixing e with $\theta^2 = \mathrm{Id}_U$

(iv) an open subset $\Omega \subset U \times U$ and a smooth map $m \colon \Omega \to U$ called *multiplication*,

such that

(a) for every $g \in U$ (e, g) and (g, e) are contained in Ω and $m(e, g) = m(g, e) = g$,

(b) for every $g \in U$ the pairs $(\theta(g), g)$ and $(g, \theta(g))$ are contained in Ω and $m(\theta(g), g) = m(g, \theta(g)) = e$,

(c) for every triple (g, h, k) of elements in G if the pairs $(g, h), (h, k), (g, m(h, k))$ and $(m(g, h), k)$ are contained in Ω then $m(g, m(h, k)) = m(m(g, h), k)$.

Clearly Properties (a),(b), and (c) are local versions of the identity law, the inverse law, and the associative law for a group. The only difference is that the domain of definition for multiplication is an open subset of $U \times U$ and the associative law only holds on a smaller open subset of $U \times U \times U$.

From now on we denote $\theta(g)$ by g^{-1}. Of course $e^{-1} = e$. We also write gh for $m(g, h)$.

The following is an elementary lemma.

Lemma 2.2. *For $g \in U$ there are open subsets $W \subset U$ containing e and $V \subset U$ containing g such that ug is defined for every $u \in W$, and vg^{-1} is defined for every $v \in V$ and the maps $u \mapsto ug$ and $v \mapsto vg^{-1}$ are inverse diffeomorphisms between W and V. Furthermore, there is an open subset $W \subset G$ such that $W^2 \times W^2 \subset \Omega$. For any $w_1, w_2, w_3 \in W$ the following two expressions $w_1(w_2w_3)$ and $(w_1w_2)w_3$ are defined and hence are equal.*

Every Lie group G *determines* the local Lie group ($U = G$ and $\Omega = G \times G$).

A *morphsm* of local Lie groups $(U', e', \theta', \Omega', m') \to (U, e, \theta, \Omega, m)$ is a smooth map $\rho\colon U' \to U$ with $\rho(e') = e$ and $\rho \times \rho|_{\Omega'}\colon \Omega' \to \Omega$ such that $\rho(\theta'(x)) = \theta(\rho(x))$ for all $x \in U'$ and $m(\rho(x), \rho(y)) = \rho(m'(x, y))$ for all $(x, y) \in \Omega'$. It is clear that these morphisms can be composed and that each object has the identity morphism. Hence we have a category of local Lie groups.

A morphism of Lie groups is a morphism of the local Lie groups they determine.

Lemma 2.3. *Any local Lie group has a Lie algebra whose underlying vector space is the tangent space T_eU. The differential at e of a morphism of local Lie groups induces a homomorphism of their Lie algebras.*

Proof. The argument that this defines a Lie algebra follows the proof in the case of a Lie group. Given $X \in T_eU$ there is a vector fields on U whose value at $g \in U$ is $g \cdot X$. (For any g multiplication by g is define on a neighborhood of e, and hence multiplication by g sends $T_eU \to T_gU$.) We call all such vector fields *left-invariant*. Then the usual argument shows that the space of these vector fields is closed under bracket and the space is identified with T_eU. Hence, there is the induced Lie algebra structure on T_eU, which is defined to be *its Lie algebra*.

The other approach to the Lie bracket also works for local Lie groups. For any $g \in U$, the element $m(ge, g^{-1}) = geg^{-1}$ is defined and hence for every $g \in U$ there is a neighborhood V of $e \in U$ such that $\rho_g(v) = gvg^{-1}$ is defined. for all $v \in V$. Thus, the differential of this map ρ_g at $v = e$ determines a map $\overline{\rho}_g\colon U \to GL(T_eU)$ and restricted to sufficiently small neighborhood of $e \in U$ this map is a local homomorphism in the sense that $\overline{\rho}_h\overline{\rho}_g = \overline{\rho}_{hg}$ for h, g sufficiently close to the identity. Define $\mathrm{ad}(X)\colon T_eU \to T_eU$ as the image of X under the differential, $d\rho_e\colon T_e(U) \to \mathrm{End}(T_eU)$. The Lie algebra of the local Lie group is then $[X, Y] = \mathrm{ad}(X)(Y)$.

The proof that these two methods define the same Lie algebra follows by the same argument as in the case of a Lie group. $\square$

Let $G^0 = (U, e, \theta, \Omega, m)$ be a Local Lie group. Let $U' \subset U$ an open subset containing e and invariant under θ and let $\Omega' \subset \Omega$ open subset containing $(\{e\} \times U') \cup (U' \times \{e\})$ and $\{g, \theta(g)\}$ for all $g \in U'$ and such that $m(\Omega') \subset U'$. Set $m' = m|_{\Omega'}$ and $\theta' = \theta|_{U'}$. Then we say that $(U', e, \theta', \Omega', m')$ is *a neighborhood of the identity in G^0*. Notice that any open subset $U' \subset U$

containing e and invariant under θ can be completed to a neighborhood of the identity of the local Lie group $(U, e, \theta, \Omega, m)$.

Corollary 2.4. *Let G be a Lie group and $U \subset \mathfrak{g}$ an open subset containing 0 and invariant under $X \mapsto -X$ and for which the restriction of* exp *to U is a diffeomorphism onto an open subset of G. Then there is a local Lie group $(U, e, \theta, \Omega, m)$ that is a neighborhood of the identity in the local Lie group determined by G.*

Viewed from the categorical perspective we have two natural transformations:

Lie Groups —–> Local Lie Groups —–> Lie Algebras

The object of the rest of this lecture and the next two (IIIA and IIIB) is to find morphisms in the opposite direction.

3 Extending local Lie Subgroups of a Lie group

Let's begin by showing how to enhance a local Lie sub group of a Lie group to a Lie group that is one-to-one immersed in G. This will give an inverse for the first morphism above but only on the subcategory of local Lie groups that can be embedded as subgroups of some Lie group.

Theorem 3.1. *Let G be a Lie group and let $(U, e, \theta, \Omega, m)$ be a local Lie sub group of G. (This means that there is a morphism of local Lie groups $(U, e, \theta, \Omega, m) \to G$ which is a locally closed embedding on U.) Then there is a Lie group N and an identification of $(U, e, \theta, \Omega, m)$ as a neighborhood of the identity in N. Furthermore, there is a one-to-one immersion of $N \to G$ whose restriction to U is the identity. The subgroup of G generated by U is an open subgroup of N. If U is connected, then this subgroup is the connected component of the identity of N.*

Proof. Let $N \subset G$ be the set of elements $g \in G$ such that gUg^{-1} contains an open neighborhood of the identity in U.

Claim 3.2. *For any $g \in N$, there is an open neighborhood V of the identity in U such that the map $V \to G$ given by $v \mapsto gvg^{-1}$ is a diffeomorphism of V onto an open subset $V' \subset U$ containing the identity.*

Proof. By the definition of N there is an open subset $W \subset U$ containing the identity with $W \subset gUg^{-1}$. Thus, $V = g^{-1}Wg \subset U$. Since conjugation by g^{-1} is a diffeomorphism of G, the map $w \mapsto g^{-1}wg$ is a smooth map from $W \to U$ whose image is V. Being the restriction of a diffeomorphism to a smooth submanifold of G, this map is one-to-one has injective differential at each point. It follows that as a map $W \to U$ it has surjective differential at each point and hence is a diffeomorphism onto an open subset of U. This shows that V is an open subset of U containing e. Conjugation by g is the inverse diffeomorphism from $V \subset U$ to $W \subset U$. □

Claim 3.3. *N is a subgroup of G.*

Proof. Suppose that $g \in N$. Then according to the previous claim there is an open neighborhood $V \subset U$ of e such that conjugation by g maps it diffeomorphically onto an open neighborhood $W \subset U$ of e. Of course, conjugation by g^{-1} takes W to V, establishing that $g^{-1} \in N$.

Now suppose that $g_1, g_2 \in N$. Let V be an open neighborhood of e in U such that conjugation by g_2^{-1} sends V to an open neighborhood $V_1 \subset U$ of e. Then $V \cap W$ is an open neighborhood of e in U and conjugation by g_2^{-1} sends this diffeomorphically onto $T = (V_1 \cap g_2^{-1} W g_2)$ which is an open subset of $g_2^{-1}(V_1)g_2 = V \subset U$ containing e. Now $g_1(g_2 T g_2^{-1})g_1^{-1}$ is an open subset of $g_1(V \cap W)$ which in turn is an open subset of $g_1 W g_1^{-1}$ which lastly is an open subset of U. This proves that $g_1 g_2 \in N$, and completes the proof that N is a subgroup o G. □

There is a neighborhood $V \subset U$ of e such that $V \times V \subset \Omega$, and given V there is a subset $W \subset V$ such that $m(W, W) \subset V$. Now we define a topology on N that makes it a smooth manifold of the dimension of U. Namely, for any $g \in N$ we define gW to be an open neighborhood of $g \in N$ with the topology and smooth structure it inherits from $W \subset U$ translated by left multiplication by g.

To show that these choices define a topology and a smooth manifold structure on N we need only show that on two-fold overlaps the smooth structures are compatible, meaning the the overlap function from one neighborhood to the other is a diffeomorphism. So let gW and $g'W$ be two smooth patches with $gW \cap g'W \neq \emptyset$. Take a point x in the intersection.Then there are $w, w' \in W$ with $x = gw = g'w'$. It follows that $g^{-1}g' = w(w')^{-1}$, and hence $g^{-1}g' \in W^2 \subset V$. This means that multiplication by $g^{-1}g' \colon W \to W$ is a multiplication in the local Lie group, and hence multiplicaton by $g^{-1}g'$ is a smooth map from $W \to U$. This smooth map carries the open subset $(g')^{-1}(g'W \cap gW) \subset W$ to the open subset $g^{-1}(gW \cap g'W) \subset W$ and is exactly the overlap transformation in one direction. The symmetric argument show that the inverse overlap function is also smooth. Since these maps are inverses of each other, each is a diffeomorphism.

This completes the proof that we have defined a smooth manifold structure on N. It has the property that the restriction of this smooth structure to $U \subset N$ agrees with the smooth structure U already has. Thus, U is a neighborhood of e in N. Notice that the inclusion map $N \to G$ is smooth immersion and is one-to-one. It follows that since G is a Hausdorff space, so is N.

Next we show that with this smooth structure N the group multiplication from G is smooth. We fix $g_1, g_2 \in$ and consider the product map $g_1W \times g_2W \to N$. By restricting to a smaller neighborhood of the identity $W' \subset W$ we can suppose that the image of multiplication $g_1W' \times g_2W'$ lies in g_1g_2W. The map is given by

$$(g_1w)(g_2w') = (g_1g_2)(g_2^{-1}wg_2))w'.$$

Since $g_2 \in N$, if we restrict to a sufficiently small neighborhood T of e in W conjugation by g_2^{-1} sends T diffeomorphically onto $T' \subset W$. Since the product $W \times W \to U$ is smooth, it follows that

$$(g_1w)g_2(w') \mapsto g_1g_2(g_2^{-1}wg_2)w'$$

is a smooth map in some neighborhood of (e, e)

Lastly, we show that $g \mapsto g^{-1}$ is a smooth map $N \to N$. We fix $g \in N$ and consider the inverse map from gW to $g^{-1}N$. The map sends gw to $g^{-1}gw^{-1}g^{-1}$. As before, since $g \in N$, restricting to a smaller neighborhood of e in W conjugation of g sends that neighborhood diffeomorphically onto another neighborhood of e in W. Then since the inverse in W is given by θ, it is also smooth, showing that $g \to g^{-1}$ is a smooth map of N to itself.

This completes the proof that N is a Lie group, that the inclusion of $N \to G$ is a one-to-one immersion of Lie groups, and that the local Lie group $(U, e, \theta, \Omega, m)$ is a neighborhood of the identity in N. □

While N may not be second countable, if U is second countable, then the subgroup of N generated by U is second countable. In particular, the connected component of the identity $N_0 \subset N$ is a second countable Lie group.

Example. Let G be a Lie group and $U = \{e\}$. It is a sub local Lie group. Then $N = G$ and the topology on N is the discrete topology. The immersion $N \to G$ is the identity map, which is a surjective, one-to-one immersion, but far from a diffeomorphism. Any time the local Lie group has a positive dimensional normalizer in G, then the Lie algebra N will have uncountably many components.

4 Lie Algebras to local Lie Groups

4.1 The Baker-Campbell-Hausdorff Formula

Constructing a local Lie group from a Lie algebra relies on the BCH formula, so we begin with that formula.

We have shown that for any Lie group G there is a local Lie group that is a neighborhood of the identity in G and whose underlying submanifold U is the diffeomorphic image of an open subset in the Lie algebra under the exponential mapping. The question naturally arises as to whether the multiplication in a local Lie group that is a sufficiently small neighborhood of the identity in G is determined by the Lie bracket (and the linear structure) on the Lie algebra. The answer is 'yes,' and in fact the multiplication for the local Lie group structure is given by the Baker-Campbel-Hausdorff formula.

One way to view the question is to consider two elements e^A and e^B in G for $A, B \in \mathfrak{g}$ sufficiently close to zero. The goal is to write the product $e^A e^B$ as $e^{H(A,B)}$ where $H(A, B)$ is a convergent power series (with some positive radius of convergence) whose n^{th} order terms are universal linear combinations of all possible brackets of A and B of order n, that is to say linear combinations of brackets of n terms each of which is either A or B.

Let us examine the first few terms in the case of $GL_n(\mathbb{R})$ to see how this would work. We write

$$e^A e^B = \sum_{n,m} \frac{A^n B^m}{n!m!} = 1 + (A + B) + (A^2/2 + AB + B^2/2) + (A^3/6 + A^2B/2 + AB^2/2 + B^3/6) + \cdots .$$

Thus, the power series for $H(A, B)$ begins

$$H(A, B) = (A + B) + \cdots .$$

Let us compute the quadratic term $Q(A, B)$ in $H(A, B)$. It must satisfy the equation

$$A^2/2 + AB + B^2/2 = (A + B)^2/2 + Q(A, B).$$

Thus,

$$Q(A, B) = AB - (AB + BA)/2 = (AB - BA)/2 = \frac{1}{2}[A, B].$$

The cubic term $C(Q,B)$ in $H(A,B)$ satisfies

$$\begin{aligned} &A^3/6 + A^2B/2 + AB^2/2 + B^3/6 = \\ =\quad & (A+B)^3/6 + [(A+B)Q(A,B) + Q(A,B)(A+B)]/2 + C(A,B) \\ =\quad & \frac{1}{6}(A^3 + A^2B + ABA + BA^2 + AB^2 + BAB + B^2A + B^3) \\ & + \frac{1}{4}[(A+B)[A,B] + [A,B](A+B)] + C(A,B). \end{aligned}$$

Cancelling terms and expanding yields:

$$\begin{aligned} \frac{1}{3}(A^2B + AB^2) =& (\frac{1}{6}(ABA + BA^2 + BAB + B^2A) \\ & + \frac{1}{4}(A^2B - ABA + BAB - B^2A + ABA - BA^2 + AB^2 - BAB) + C(A,B). \end{aligned}$$

Collecting terms gives

$$\frac{1}{12}(AB^2 + A^2B + B^2A + BA^2) - \frac{1}{6}(BAB + ABA) = C(A,B).$$

Thus,

$$C(A,B) = \frac{1}{12}([A,[A,B]] + [B,[B,A]]).$$

Theorem 4.1. *(Baker-Campbell-Hausdorff Formula) Let L be the free Lie algebra generated by X and Y. There is a formal infinite sum $H(X,Y)$ in two variables where the n^{th} term in the sum is a linear combination of the Lie brackets of order n of X and Y*

$$[Z_1,[Z_2,\cdots.,[Z_{n-1},Z_n]]\cdots]$$

where the Z_i range over X and Y, such that there is an equality of formal power series

$$\log(\exp(X)\exp(Y) = H(X,Y).$$

For any finite dimensional real Lie algebra L, fixing a positive definite form Q on L there is $r > 0$ and defining $U \subset L$ by $U = \{X \in L \mid Q(X) < r\}$. The power series $H(A,B)$ converges absolutely for $(A,B) \in U \times U$ and defines an analytic function $H \colon U \times U \to L$. The open set U is invariant under $X \mapsto -X$. Let $\Omega \subset U \times U$ be $H^{-1}(U)$. Defining $\theta(A) = -A$ and $m(A,B) = H(A,B)$, makes $(U,0,\theta,\Omega,m)$ is a local Lie group. If $L = \mathfrak{g}$ for a Lie group G, and possibly replacing r by r' with $0 < r' < r$ so that $\exp|_U$ is a diffeomorphism onto an open subset of G, the restriction of the exponential mapping to U defines an embedding of $(U,0,\theta,\Omega,m)$ onto a local Lie sub group of G that neighborhood of the identity.

We call any such local Lie group a defined on an open set $U \subset L$ of 0 sufficiently small so that the BCH series converges and invariant under $X \mapsto -X$ *a local Lie group determined by the Lie algebra* L. Any two such have a common neighborhood of 0.

There is an explicit formula due to Hausdorff, and even a recursive formula for the coefficients. But the actually coefficients are not important. The only important thing is that a convergent series exists and has a positive radius of convergence. The proof that a series exists uses the Poincaré-Birkhoff-Witt Theorem. The convergence is a direct computation that we leave to the exercises.