Lie Groups: Fall, 2022
Lecture II
Real and Complex Lie Groups, Lie Algebras, and the Adjoint Actions

September 13, 2022

1 Real Lie Groups

1.1 Basic Definitions

Definition 1.1. A *(real) Lie group* consists of a smooth (finite dimensional) manifold G together with smooth maps

$$m\colon G \times G \to G$$

$$\iota\colon G \to G.$$

The map m is called the *product* and is usually written by juxtaposition. The map ι is the *inverse map* sending every element to its inverse. These are required to define a group structure on G, meaning that m is associative, there is an element $e \in M$ such that for all $g \in M$, $m(\iota(g), g) = m(g, \iota(g)) = e$, and $m(e, g) = m(g, e) = g$ for all $g \in M$. A map of Lie groups $\rho\colon H \to G$ is a *morphism of Lie groups* if it is a group homomorphism and a smooth map. The category of Lie groups has as objects Lie groups and morphisms as just defined. An isomorphism of Lie groups is a diffeomorphism between the underlying manifolds that is also a group isomorphism.

Definition 1.2. Let M be a smooth manifold. A *smooth submanifold* is a subset $N \subset$M with the property that for each $n \in N$ there is a local coordinate system $(x^1, \ldots, x^k)$ defined on an open set U containing the point n such that $N \cap U$ is given by the equations $\{x^{r+1} = \cdots = x^k = 0\}$. Then N inherits a unique smooth structure such that the inclusion $N \to M$

is a smooth map. Such a map is called *a smooth embedding*. [Since we are working exclusively in the smooth category we shall drop the adjective *smooth* from the terminology both for submanifolds and embedding. It is imiplicit.]

Notice that if N is a submanifold of M then it is a closed subset of M. There is a converse to this. Suppose that $\varphi\colon N \to M$ is an immersion (injective differential at every point) and is a one-to-one map. Then the image $\varphi(N)$ is a submanifold of M if and only if it is a closed subset. An example showing that the image is not automatically closed is given by the map

$$\mathbb{R}^1 \xrightarrow{f} \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

where $f(t) = (t, \pi t)$.

Definition 1.3. Let G be a Lie group. A *Lie subgroup* $H \subset G$ is a smooth submanifold H of G that is closed under the product and inverses and contains the identity element[1].

The terminology is justified by the following lemma.

Lemma 1.4. *If $H \subset G$ is a sub-Lie group, then the restriction of the product and inverse of G to H give H the structure of a Lie group and the inclusion $H \subset G$ is a morphism of Lie groups. Furthermore, the space of left cosets G/H has the structure of a smooth manifold in such a way that the projection $G \to G/H$ is a submersion (i.e., has surjective differential at every point).*

Proof. Since H is closed under product and inverses and contains the identity, the restriction of the group structure maps from G to H define the structure of a group on H. We need only see that the product and the inverse are smooth maps of H. But they are smooth maps of G and H is a smooth submanifold invariant under the maps. Hence, the restriction of the maps to H are smooth. The inclusion $H \subset G$ is a smooth map and a group homomorphism and hence, by definition a morphism of Lie groups.

Fix a local coordinate system for G centered at the identity and a ball B about the origin in the coordinate system such that $H \cap B = \{x^{r+1} = \cdots = x^k = 0\} \cap B$. Let S be the intersection of B with the coordinate plane spanned by the unit vectors in the first r directions. (We call any intersection of a ball centered at the origin with the coordinate space spanned by

[1]In the literature one sometimes finds the more general notion of sub Lie group where the submanifold is not required to be closed, just to be one-to-one immersed.

$(x^1, \ldots, x^r)$ a *slice*.) Then $T_eG = T_eH \oplus T_eS$. It follows that the differential of the map $\mu\colon H \times S \to G$ sending $(h, s) \to hs$ is an isomorphism at $(e, 0)$. Hence, there is a sub-ball $B' \subset B$ centered at the origin, such that setting $S' = S \cap B'$, there is an open neighborhood U of $e \in H$ such that the restriction of μ to a map $\mu\colon U \times S' \to G$ is a diffeomorphism onto an open subset of G. By H-equivariances, the restriction of $H \times S' \to G$ is a local diffeomorphism onto an open neighborhood of H in G.

We claim that possibly after shrinking B' to a smaller open ball centered at the origin and consequently shrinking the slice S' to a smaller 'slice,', the map $H \times S' \to G$ is a diffeomorphism onto an open subset. For the restriction of the map to a smaller slice $H \times S'' \to G$ to be a diffeomorphism, it suffices to show that it is one-to -one. If there is no such S'', then there are sequences $\{s_i\}_{i<\infty}$ and $\{s'_i\}_{i<\infty}$ in S' both converging to zero and sequences $\{h_i\}_{i<\infty}$ and $\{h'_i\}_{i<\infty}$ in H such that for every i we have $h_is_i = h'_is'_i$ yet $(h_i, s_i) \neq (h'_i, s'_i)$. Multiplying by h_i^{-1}, we can assume that $h_i = e$ for all i.. Hence $h'_is'_i \mapsto e$ and $s'_i \mapsto e$, so that h'_i converges to e. Since $H \times S' \to G$ is a local diffeomorphism near $(e, 0)$, it follows that for all i sufficiently large $h'_i = e$ and $s'_i = s_i$. This is a contradiction.

We have now shown that for a sufficiently small slice S the map $H \times S \to G$ is a diffeomorphism onto an open subset. Thus, S is a coordinate patch for G/H near the identity coset. Pushing these local coordinates around by $g \in G$ gives coordinate patches covering G/H. It is clear that the overlap of two of these patches is smooth and the projection map $G \to G/H$ is a smooth submersion.

Lastly, we need to show the quotient G/H is a Hausdorff space. That is to say we need to show that a sequence in the quotient space has at most one limit point. If not then there is a sequence $\{s_i\}_i$ in G converging to some $g_0 \in G$ and elements $h_i \in H$ such that the sequence h_is_i converges to a point $g_1 \in G$ with the property that g_0 and g_1 are not in the same H-orbit.

If a sequence $\{s_i\}_i$ converges to $s \in g_0H$. Then $g_0^{-1}s_i$ converges to the identity and hence for all i sufficiently large $g_0^{-1}s_i$ is contained in the a neighborhood of the form $H \times S \subset G$ (where the neighborhood is invariant under the action of H and the action is $h(h', s) = (hh', s)$. It follows that the H-orbits s_iH converge to H. Multiplying by g_0 we see that assuming that h_is_i converges to a point of $g_0^{-1}g_1 \in G$ implies that the h_i converge in to some $\overline{h} \in H$ and hence $g_1 = \overline{h}g_0$, showing that g_0 and g_1 are in the same H-orbit. $\square$

Definition 1.5. A sub Lie group $K \subset G$ is said to be *normal* if K is a normal subgroup of G in the usual group theoretic sense.

Lemma 1.6. *If $K \subset G$ is a normal Lie subgroup, then the space of left cosets G/K has the structure of a Lie group such that the projection $G \to G/K$ is a homomorphism of Lie groups.*

Proof. Since K is a normal subgroup of G, the group structure on G induces a group structure on G/K. We have just seen that G/K is a smooth manifold and that the projection is a smooth map and a group homomorphism. It remains only to show that the structure maps for the group structure on G/K are smooth. Let us consider the multiplication map $\mu\colon G/K \times G/K \to G/K$. Fix points $x, y \in G/K$. Left these to points $\widetilde{x}, \widetilde{y} \in G$ and let $S_{\widetilde{x}}, S_{\widetilde{y}}$ be slices from the projection mapping $G \to G/K$ at $\widetilde{x}$ and $\widetilde{y}$, respectively. Let $S_{\widetilde{x}\widetilde{y}}$ be a slice for the projection $G \to G/K$ at $\widetilde{x}\widetilde{y}$. Choosing $S_{\widetilde{x}}$ and $S_{\widetilde{y}}$ sufficiently small, we can assume that the image of the product $\mu(S_{\widetilde{x}} \times S_{\widetilde{y}})$ is contained in $K \times S_{\widetilde{x}\widetilde{y}} \subset G$. It is a smooth map. Thus, the composition

$$S_{\widetilde{x}} \times S_{\widetilde{y}} \xrightarrow{\mu} K \times S_{\widetilde{x}\widetilde{y}} \xrightarrow{\pi_2} S_{\widetilde{x}\widetilde{y}}$$

is also smooth. This is the restriction of the multiplication map for the quotient to $S_{\widetilde{x}} \times S_{\widetilde{y}}$.

The argument for the inverse is similar. □

There is an analogue of the first part of Lemma 1.4

Lemma 1.7. *Let G be a Lie group. Suppose that H is a smooth manifold and $\varphi\colon H \to G$ is a one-to one smooth immersion whose image is a subgroup of G. Then there is a unique Lie group structure on H so that φ is a homomorphism of Lie groups.*

Proof. Since H is a smooth manifold and a group, we need only show that group multiplication and inverse are smooth maps. Let $(h, h') \in H \times H$. There there are neighborhoods U, U' and V of h, h' and hh', respectively, such that $\varphi\colon U \to G$ and $\varphi\colon U' \to G$ and $\varphi\colon V \to G$ are embeddings onto smooth (locally closed) submanifolds. Taking U and U' sufficiently small we can arrange that the product in G maps $U \times U' \to V$. Since the group multiplication of G is smooth the composition $U \times U' \to V \subset G$ is smooth, and since V is a locally closed smooth submanifold of G, this implies that $U \times U' \to V$ is smooth.

The argument for the inverse map is analogous. □

Notice that in this case there is no reasonable manifold structure on G/H. Indeed, in general the quotient space is not Hausdorff.

1.2 Some Examples

Here are a few examples of Lie groups. The real line $\mathbb{R}$ with m being addition and $\iota(x) = -x$ is a Lie Group, the *additive group over* $\mathbb{R}$. The unit circle in the complex plane with product being product of complex numbers and ι being inverse of complex numbers is a Lie group. Let V be a finite dimensional real vector space. Then $GL(V)$ is a Lie group under matrix multiplication and matrix inverse. $SL(V)$ the subgroup of $GL(V)$ of matrices of determinant 1 is a subgroup. (Give a definition of the trace which makes no reference to a basis.)

Let Q be a non-degenerate quadratic form on a finite dimensional real vector space V. W define $O(Q)$, the *orthogonal group of* Q, to be the subgroup of $GL(V)$ that leaves Q invariant in the sense that $A \in GL(V)$ is in $O(Q)$ if and only if $Q(Av) = Q(v)$ for all $v \in V$. Check that $O(Q)$ is a smooth submanifold of $GL(V)$ that closed under the product and taking inverses and contains the identity. Applying the above lemma, we see that it is a sub-Lie group of $GL(V)$ and hence is a Lie Group in its own right.

The example $O(n)$ is the orthogonal group of the standard Euclidean inner product on $\mathbb{R}^n$. The group $SO(n)$ is the subgroup of $O(n)$ of matrices of determinant 1. Show that $SO(n)$ is the component of the identity of $O(n)$.

If G_1 and G_2 are Lie groups then the product smooth manifold $G_1 \times G_2$ is naturally a Lie group under the product operations. This is a categorical product in the category of Lie Groups. Furthermore, $G_1 \times \{e\}$ and $\{e\} \times G_2$ are sub lie groups of $G_1 \times G_2$.

1.3 Some Counter-Examples

Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The translation structure on $\mathbb{R}^2$ induces an Abelian group structure on T^2 that makes it a compact lie group. Any sub-Lie group $H \subset T^2$ is a closed subset of T^2 and hence is compact. As a result every connected sub lie groups of T^2 is isomorphic to one of T^2, S^1,$\{e\}$. If $\mathbb{R} \subset \mathbb{R}^2$ is a line through the origin in an irrational direction, then it induces an injective map $\mathbb{R}^1 \to T^2$ map os Lie groups whose image is not compact and hence not a sub Lie group.

Notice that there is a quotient space $T^2/\mathbb{R}^1$ inherits a group structure and also is is locally isomorphic to $\mathbb{R}^1$ with local coordinates in which the group structure is smooth. But the quotient is not a Lie group since it is not Hausdorff.

There are similar examples in higher dimensional tori of all possible codimensions ≥ 1.

The examples show that in general if $\rho\colon H \to G$ is a map of Lie groups, then the image is not necessarily a sub Lie group of G.

1.4 Actions and Representations

Definition 1.8. Let M be a smooth manifold and G a Lie group. A smooth action of G on M is a smooth map $f\colon G \times M \to M$ that is an action of the group G on the set M. Thus, each $g \in G$ acts by a diffeomorphism of M and these vary smoothly with $g \in G$. Equivalently, we have a smooth map $G \to Diff(M)$. We often leave the term *smooth* implicit and simply use *action* to mean smooth action.

Definition 1.9. A linear representation is a smooth action $G \times V \to V$ where V is a topological vector space (usually either finite dimensional or a Hilbert space) and the action of each $g \in G$ is by continuous linear isomorphisms. In the finite dimensional case, a linear representation is the same thing as a smooth map $G \to GL(V)$. In the case of a complex-linear action on a complex Hilbert space H an action $G \times H \to H$ is *unitary* if $\langle gx, gy \rangle = \langle x, y \rangle$ for all $g \in G$ and all $x, y \in H$. That is to say the action is given by a continuous group homomorphism from G to $U(H)$ the infinite dimensional group of unitary transformations of H.

2 Complex Lie groups

2.1 Basic Definitions

There is another category of Lie Groups.

Definition 2.1. A *complex Lie Group* consists of a finite dimensional complex manifold G together with structure maps – a product and an inverse – that are maps of complex manifolds and make G a group. A morphism of complex Lie groups $H \to G$ is a holomorphic map that is also a group homomorphism. These are the objects and morphisms, respectively, of the category of complex Lie groups.

Any linear algebraic group over $\mathbb{C}$ is automatically a complex Lie Group.

As in the real case, we have:

Lemma 2.2. *If G is a complex Lie Group and $H \subset G$ is a complex submanifold containing the identity element of G and closed under the product operation and the inverse map, then H together with the restriction to H of these structure maps is a complex Lie Group.*

2.2 Examples of complex Lie Groups

There are complex analogues of all the real Lie Groups mentioned above. The additive group of complex numbers and the multiplicative group $\mathbb{C}^*$ on non-zero complex numbers are both linear algebraic groups over $\mathbb{C}$ and hence complex manifolds. If V is a finite dimensional complex vector space then its linear automorphisms form a complex Lie Group. Of course, we can assume that V is isomorphic to $\mathbb{C}^n$ for some $n \geq 0$. Thus, for some $n \geq 0$ the complex Lie group $GL(V)$ is isomorphic to the complex Lie Group $GL(n, \mathbb{C})$. the group of invertible $n \times n$ complex matrices. The product is matrix multiplication and the inverse is the matrix inverse. The group is an open subset of the complex vector space $M(n \times n, \mathbb{C})$ of complex $n \times n$ matrices. In fact, being the complement of the divisor where $\det = 0$, $GL(n, \mathbb{C})$ is a Zariski open set and is a linear algebraic group over $\mathbb{C}$. We also have $SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ of matrices of determinant 1 also a linear algebraic group over $\mathbb{C}$, and hence a complex Lie Group. For any non-degenerate complex quadratic form Q on $\mathbb{C}^n$ we have its complex orthogonal group, defined as in the real case. This also is a linear algebraic group over $\mathbb{C}$ and hence a complex Lie group. Similarly, for a non-degenerate, skew symmetric, complex bilinear form on $\mathbb{C}^n$ we have the complex symplectic group, again a linear algebraic group over $\mathbb{C}$, and hence a complex Lie group.

Consider a maximal lattice $\Lambda \subset \mathbb{C}$. By definition Λ is generated by two elements that are linearly independent over $\mathbb{R}$. The quotient $\mathbb{C}/\Lambda$ is a compact complex curve diffeomorphic to $S^1 \times S^1$. Addition on $\mathbb{C}$ induces a group structure on $\mathbb{C}/\Lambda$ that makes it a complex Lie group.

2.3 Actions and Representations

Definition 2.3. Let G be a complex Lie group and M a holomorphic manifold. An *holomorphic action* of G on M is a holomorphic map $G \times M \to M$ that defines in the group theoretic sense an action of G on M. A *complex linear representation* is a holomorphic action $G \times V \to V$ on a complex vector space with the property that every $g \in G$ acs by a complex linear transformation.

A complex linear representation of a complex Lie group G on a complex vector space $V_{\mathbb{C}}$ is the same thing as a homomorphism of complex Lie groups $G \to GL(V_{\mathbb{C}})$.

3 The Adjoint Action and the Lie Algebra

Let G be a real Lie Group. There is a natural action of G (the first copy) on itself (the second copy) by conjugation:

$$\mathrm{Ad}_G\colon G \times G \to G$$

defined by $\mathrm{Ad}_G(g, g') = gg'g^{-1}$. This is a left action of G on itself, called the *adjoint* action. When G is clear from the context we denote this adjoint map simply as Ad. The action is smooth and In the case of a complex Lie group the action is holomorphic.

The adjoint action fixes $e \in G$ and hence differentiating at the identity of the second variable gives an induced linear action $\mathrm{Ad}_G\colon G \times T_eG \to T_eG$. We use the standard notation and denote T_eG by $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ is a representation of G as linear automorphisms of $\mathfrak{g}$. That is to say we have a linear representation which is a morphism of Lie groups

$$G \xrightarrow{\mathrm{Ad}_G} GL(\mathfrak{g}).$$

In the case of a complex Lie group this is a complex linear representation of G on the complex vector space $\mathfrak{g}$, i.e., it determines a holomorphic map $G \to GL_C(\mathfrak{g})$. We can differentiate this Lie group morphism at the identity of G and obtain a linear map from $\mathfrak{g}$ to the endomorphism ring of $\mathfrak{g}$

$$\mathrm{ad}_G\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}).$$

Example. Let $G = GL(n, \mathbb{R})$. Then $\mathfrak{g}$, which is denoted $\mathfrak{g}l(n, \mathbb{R})$ in this case, is the vector space $M(n \times n, \mathbb{R})$ of $n \times n$ real matrices. Differentiating the conjugation action of G on itself at the identity (in the second variable) produces the usual conjugation action of G on $\mathfrak{g}l(n, \mathbb{R}) = M(n \times n, \mathbb{R})$. We compute the differential of this action at $e \in G$. Let $A(t)$ be a one-parameter family in $GL(n, \mathbb{R})$ with $A(0) = \mathrm{Id}$ and denote by $A_0 \in M(n \times n, \mathbb{R})$ the derivative of this family at $t = 0$. Then $(A^{-1})'(0) = -A_0$. Fix $B \in M(n \times n, \mathbb{R})$. Then we have

$$\frac{d(A(t)BA(t)^{-1})}{dt}\Big|_{t=0} = A_0B - BA_0.$$

Thus,

$$\mathrm{ad}\colon \mathfrak{g}l(n, \mathbb{R}) \to \mathrm{End}(\mathfrak{g}l(n, \mathbb{R}))$$

sends $A_0 \in M(n \times n, \mathbb{R})$ to the endomorphism $[A_0, \cdot]$, where $[A, B] = AB - BA$ is the usual Lie bracket of $n \times n$ matrices. Said another way $\mathrm{ad}_G(A)(B) = [A, B]$, the Lie algebra bracket of $\mathfrak{g}l(n, \mathbb{R})$. One also writes $\mathrm{ad}_G(A, B)$ for $\mathrm{ad}_G(A)(B)$.

Proposition 3.1. *Suppose that $H \subset GL(n,\mathbb{R})$ is a sub-Lie group. Let $\mathfrak{h} \subset M(n\times n,\mathbb{R})$ be the tangent space to H at the identity. Then $\mathfrak{h}$ is closed under Lie bracket of matrices; and the resulting action $\mathfrak{h} \to \mathrm{End}(\mathfrak{h})$ sends $X \in \mathfrak{h}$ to the endomorphism of $\mathfrak{h}$ given by $[X,\cdot]$, so that the Lie algebra structure on $\mathfrak{h}$ is given by the restriction of the Lie algebra structure on $\mathfrak{g}l(n,\mathbb{R})$.*

Proof. Consider the restriction of $\mathrm{Ad}_{GL(n,\mathbb{R})}\colon GL(n,\mathbb{R})\times\mathfrak{g}l(n,\mathbb{R}) \to \mathfrak{g}l(n,\mathbb{R})$ to $H \subset GL(n,\mathbb{R})$ is $\mathrm{Ad}_H\colon \mathfrak{g}l(n,\mathbb{R}) \to \mathfrak{g}l(n,\mathbb{R})$. This restriction leaves $\mathfrak{h} \subset \mathfrak{g}l(n,\mathbb{R})$ invariant and this restriction is $\mathrm{Ad}_H\colon H \times \mathfrak{h} \to \mathfrak{h}$. Hence, the restriction of $\mathrm{ad}_{\mathfrak{g}l(n,\mathbb{R})}\colon \mathfrak{g}l(n,\mathbb{R}) \times \mathfrak{g}l(n,\mathbb{R}) \to \mathfrak{g}l(n,\mathbb{R})$ to $\mathfrak{h} \times \mathfrak{H}$ is $\mathrm{ad}_{\mathfrak{h}}$. $\square$

Definition 3.2. The subspace $\mathfrak{h} \subset \mathfrak{g}l(n,\mathbb{R})$ together with the induced Lie bracket is the *Lie algebra of H*.

The adjoint action of $GL(n,\mathbb{C})$ on itself is as automorphisms of the complex Lie group varying holomorphically with $g \in GL(n,\mathbb{C})$. In particular, the Lie algebra $\mathfrak{g}l(n,\mathbb{C}) = M(n\times n,\mathbb{C})$ is a complex vector space and the adjoint action of $GL(n,\mathbb{C})$ on $\mathfrak{g}l(n,\mathbb{C})$ is the conjugation action of $GL(n\mathbb{C})$ on $M(n\times n,\mathbb{C})$. Thus, the Lie algebra $\mathfrak{g}l(n,\mathbb{C})$ of $GL(n,\mathbb{C})$ is the space $M(n\times n,\mathbb{C})$ with the bracket being given by $[A,B] = AB - BA$, this time of complex matrices..

3.1 The Lie Algebra of a General Lie Group

We have been able to evaluate the Lie Algebra for $GL(n,\mathbb{R})$ and for all its Lie subgroups. In particular, we have shown that for such groups H the map $\mathfrak{h}\times\mathfrak{h} \to \mathfrak{h}$ given by $(X,Y) \mapsto \mathrm{ad}(X)(Y)$ is a Lie usual Lie bracket of matrices. We turn now to extending this to arbitrary Lie groups.

3.1.1 Vector Fields

Recall that the (infinite dimensional) space of smooth vector fields on a manifold has a Lie bracket. If X and Y are vector fields, then their bracket $[X,Y]$ is defined by giving its value on a general function f by $[X,Y](f) = X(Y(f)) - Y(X(f))$. As we checked in the last lecture by direct computation, the second order derivative terms in $X(Y(f))$ cancel those of $Y(X(f))$ (basically this is equality of cross partials) so that the bracket is again a vector field. Then we can invoke the fact that the bracket is written $XY - YX$ in the associative algebra of all differential operators and hence this bracket defines a Lie algebra.

Definition 3.3. A vector field χ on G is *left-invariant* if for each $g \in G$ and each $x \in G$, $D(g\cdot)(\chi(x)) = \chi(gx)$.

Lemma 3.4. *1. Given $X \in \mathfrak{g}$ there is a unique left-invariant vector field χ_X whose value at the identity is X.*
2. If X and Y are left-invariant vector fields, then so is $[X,Y]$.

Proof. If χ is a left-inavariant vector field then $\chi(g) = D(g\cdot)\chi(e)$. This proves the uniqueness of a left-invariant vector field with a given value at the identity. Since the action $G \times TG \to TG$ given by defining the action of g to be $D(g\cdot)$ is a smooth map, for any $X \in \mathfrak{g}$, the formula $\chi(g) = D(g\cdot)X$ defines a smooth vector field, proving the existence.

Suppose that X and Y are left-invariant vector fields. Since $g\cdot$ is a diffeomorphism it commutes with the Lie bracket of vector fields. Thus, $g \cdot [X,Y] = [g \cdot X, g \cdot Y]$. □

The left-invariant vector fields on a Lie group G form a finite dimensional Lie algebra. Associating to each such vector field its value at the identity element of the group gives a linear isomorphism between the left-invariant vector fields and $\mathfrak{g}$. Transferring the Lie algebra structure from the space of left invariant vector fields to $\mathfrak{g}$ defines a Lie algebra structure on $\mathfrak{g}$. This is the *Lie algebra* of G the symbol $\mathfrak{g}$ denotes this Lie algebra structure on T_eG. If G is a complex Lie group this process defines a complex Lie algebra structure on $\mathfrak{g}$.

We now have two definitions of the Lie algebra associated to a sub Lie group of $GL(n,\mathbb{R})$: one coming from the Lie algebra associated to the associative mutltplcation of matrices and the other the Lie bracket of left-invariant vector fields. We have shown that the first Lie bracket agrees with $\mathrm{ad}_G(X)(Y)$. Thus, to show that the two definitions agree, it suffices to show that the second is also given by the same formula using ad_G. That is the consequence of the following proposition.

Proposition 3.5. *For $X, Y \in \mathfrak{g}$ we have $ad(X)(Y) = [X,Y]$, the bracket coming from the Lie bracket of the left-invariant extensions of X and Y. In particular, $X \otimes Y \mapsto \mathrm{ad}(X)(Y)$ defines a Lie algebra structure on $\mathfrak{g}$.*

Proof. Let G be a Lie group with Lie Algebra $\mathfrak{g}$. Let X and Y be elements of $\mathfrak{g}$. Extend them to left-invariant vector fields on G, denoted $\widetilde{X}$ and $\widetilde{Y}$, respectively. For each $g \in G$, let $\xi(s)$ be the integral curve for $\widetilde{X}$ though e and let $\varphi(t)$ be the integral curve for $\widetilde{Y}$ through e. Then $\varphi'(t) = \varphi(t)Y$ and $\xi'(s) = \xi(s)X$. We define a map T from a neighborhood of the origin in (s,t)-space to G by setting $T(s,t)=\varphi(t)\xi(s)$. Then

$$(\partial T/\partial s)(s,t)=\varphi(t)\xi'(s)=\varphi(t)\xi(s)X=T(s,t)X,$$

so that $(\partial T/\partial s)$ is the restriction of $\widetilde{X}$ to the surface $T(s,t)$.

On the other hand,

$$(\partial T/\partial t)(s,t)=\varphi'(t)\xi(s)=\varphi(t)Y\xi(s).$$

Thus, we have

$$\widetilde{Y}|_{T(0,t)}=(\partial T/\partial t)(0,t)$$
$$\widetilde{Y}|_{T(s,0)}=\mathrm{Ad}(\xi(s))\left(\frac{\partial T}{\partial t}(s,0)\right)$$

It follows immediately that on this surface we have $\widetilde{X}|_{T(s,t)}=(\partial T/\partial s)(s,t)$, $\widetilde{Y}|_{T(0,t)}=(\partial T/\partial t)(0,t)$.

Thus,

$$\begin{aligned}\widetilde{Y}(\widetilde{X}(f))(0,0)=&(\partial T/\partial t)_{t=0}(\partial T/\partial s)_{(0,t)}(f)\\ =&(\partial/\partial t)_{t=0}(\partial/\partial s)_{(0,t)}(T^*f)=\frac{\partial^2(T^*f)}{\partial t\partial s}(0,0)\end{aligned}$$

and

$$\widetilde{X}(\widetilde{Y})(f)(0,0)=(\partial T/\partial s))_{s=0}[\mathrm{Ad}(\xi(s))(\partial T/\partial t)_{(s,0)}].$$

Differentiating $\mathrm{Ad}(\xi(s))$ using $\partial T/\partial s$ at $s=0$ yields $\mathrm{ad}(X)$ so that since $(\partial T/\partial t)(0,0)=Y$ differentiation of this term in the entire expression yields $\mathrm{ad}(X)(Y)(f)(0,0)$. Since $\mathrm{Ad}(\xi(0))=\mathrm{Id}$, the other term in the differentiation using $\partial T/\partial s$ yields

$$(\partial T/\partial s)(\partial T/\partial t)(f)(0,0)=\frac{\partial^2(T^*f)}{\partial s\partial t}(0,0).$$

The equality of the cross partials then implies that

$$\widetilde{X}(\widetilde{Y}(f))(0,0)-\widetilde{Y}(\widetilde{X}(f))(0,0)=\mathrm{ad}(X)(Y)(f),$$

showing that $[\widetilde{X},\widetilde{Y}](0,0)=\mathrm{ad}(X)(Y)$, as claimed. $\square$

For a complex Lie group G, its Lie algebra $\mathfrak{g}$ is a complex vector space and $\mathrm{Ad}_{\mathrm{G}}\colon G\times\mathfrak{g}\to\mathfrak{g}$ is a holomorphic map. The same arguments show that $\mathrm{ad}_G\colon\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a complex bilinear pairing producing a complex Lie algebra structure on $\mathfrak{g}$ and in the case when $G\subset gl(n,\mathbb{C})$ this complex Lie algebra structure agrees with the one coming from Lie bracket of complex matrices.

3.2 Naturalness of Lie Algebra of a Lie Group

Proposition 3.6. *Let $\varphi\colon H \to G$ be a Lie group homomorphism. Then its differential at the identity $d\varphi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a map of Lie algebras, i.e., a linear map commuting with the Lie bracket operations.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} H \times H & \xrightarrow{\mathrm{Ad}_H} & H \\ {\scriptstyle \varphi\times\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ G \times G & \xrightarrow{\mathrm{Ad}_G} & G. \end{array}$$

Differentiating at $e \in H$ in the second factor produces a commutative diagram

$$\begin{array}{ccc} H \times \mathfrak{h} & \xrightarrow{\mathrm{Ad}_H} & \mathfrak{h} \\ {\scriptstyle \varphi\times d\varphi_e}\big\downarrow & & \big\downarrow{\scriptstyle d\varphi_e} \\ G \times \mathfrak{g} & \xrightarrow{\mathrm{Ad}_G} & \mathfrak{g}. \end{array}$$

Lastly, differentiating at the identity in the first variable gives a commutative diagram

$$\begin{array}{ccc} \mathfrak{h} \times \mathfrak{h} & \xrightarrow{\mathrm{ad}_H} & \mathfrak{h} \\ {\scriptstyle d\varphi_e\times d\varphi_e}\big\downarrow & & \big\downarrow{\scriptstyle d\varphi_e} \\ \mathfrak{g} \times \mathfrak{g} & \xrightarrow{\mathrm{ad}_G} & \mathfrak{g}. \end{array}$$

This diagram says that for $X, Y \in \mathfrak{h}$, we have

$$d\varphi_e(\mathrm{ad}_H(X)(Y)) = \mathrm{ad}_G(d\varphi_e(X), d\varphi_e(Y)).$$

By definition of the bracket, this translates to

$$d\varphi_e([X,Y]) = [d\varphi_e(X), d\varphi_e(Y)],$$

which is the statement that $d\varphi_e$ commutes withs the Lie bracket operations. $\square$

Corollary 3.7. *Let V be a finite dimensional real vector space and suppose that $\rho\colon G \to GL(V)$ is a homomorphism of Lie groups. Said another way, suppose that we have a smooth action $G \times V \to V$ where each $g \in G$ acts by a linear transformation. Taking the differential ρ at the identity in G determines a map $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$. This map is a homomorphism of Lie Algebras, i.e., it is a linear map sending the Lie bracket for $\mathfrak{g}$ to the bracket of endomorphisms, which is $[A,B] = A \circ B - B \circ A$.*

Proposition 3.8. *Given a smooth action $f\colon G \times M \to M$ for each element $X \in \mathfrak{g}$ we set $\mathcal{X}$ equal to the vector field whose value at $m \in M$ is $Df_{(e,m)}(X)$. This map is an anti-homomorphism of Lie algebras, in the sense that $[X,Y] \mapsto -[\mathcal{X},\mathcal{Y}]$.*

Proof. Denote the association $X \mapsto \mathcal{X}$ by $X \mapsto \psi(X)$.

First let us consider the case when $G \times M \to M$ is a free action. Fix $x \in M$ and consider the map $\varphi_x\colon G \to M$ defined by $g \mapsto gx$. For $X \in \mathfrak{g}$ the vecgtor field $\psi(X)$ is tangent to the orbit Gx and its value at gx is $(d\varphi_x)_g(Xg)$. Thus φ_x maps the right-invariant vector field Xg to the restriction to Gx of $\psi(X)$. It follows that along the orbit Gx the bracket is given by $[\psi(X),\psi(Y)] = \psi([Xg,Yg](e))$.

Claim 3.9. *The bracket of right-invariant vector fields is right-invariant and hence defines a Lie algebra structure on $\mathfrak{g}$. This is the opposite Lie algebra from the one determined by left-invariant vector fields.*

This claim is left as an exercise.

Applying this claim we see that $[\psi(X),\psi(Y)] = \psi([Xg,Yg](e)) = -\psi[X,Y]$, proving the result when the action is free.

For a general action $G \times M \to M$ we form the action $G \times (G \times M) \to (G \times M)$ given as the product of the left action of G on itself and the given action of G on M. This is a free action, so that it for this action $[\psi(X),\psi(Y)] = -\psi([X,Y])$. Projecting the action and this equation from $G \times M$ to M gives the result in the general case. $\square$

There is also the complex analogues of these results.

Proposition 3.10. *Let $\varphi\colon H \to G$ be a homomorphism of complex Lie groups. The $d\varphi_e\colon \mathfrak{h} \to \mathfrak{g}$ is a morphism of complex Lie algebras.*

Corollary 3.11. *Let V be a finite dimensional complex vector space, let G be a complex Lie group and let $G \times V \to V$ be a complex linear representation in the sense that $\rho\colon G \to GL(V)$ is a map of complex Lie groups. Then the differential of ρ at the identity, $d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex linear map sending the Lie bracket of $\mathfrak{g}$ to the bracket of complex linear endomorphisms given by $[A,B] = A \circ B - B \circ A$.*