Lie Groups: Fall, 2022
Lecture I

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1 Lie Groups

We begin with the fundamental definitions for this course.

Definition 1.1. A *Lie group* is a smooth finite dimensional manifold G with two structure maps, which are required to be smooth maps, $m\colon G \times G \to G$ and $\iota\colon G \to G$, together with an element $e \in G$. These structure maps define a group structure on G with m as a product, e as the identity element, and ι as the map $g \mapsto g^{-1}$.

If G is a complex manifold and the structure maps m and ι are holomorphic, then G is a *complex Lie Group.*

If k is a field, we denote by $GL(n.k)$ the group of invertible $n \times n$ matrices with entires in k. It is an affine variety defined over K whose affine coordinates are the entries of the matrix and the inverse of the determinant of the matrix. If $G \subset GL(n, k)$ is an algebraic variety, i.e., is the zero locus of a finite set of k-polynomial functions in the affine coordinates of $GL(n, k)$, and if G is closed under matrix multiplication and inverses and contains the identity element, then G is a *linear algebraic group over k.*

Thus, we have defined objects of three different categories of groups. The morphisms for these categories are smooth group homomorphisms, holomorphic group homomorphisms, algebraic (i.e., polynomial) group homomorphisms., respectively.

There is one technical issue in the definition of Lie groups and complex Lie groups; namely what we mean by a manifold. There are two conditions that are optional in the definition of a manifold: Hausdorff and 2^{nd} countable (which means that there is a countable basis for the topology). Usually, manifolds are assumed to be Hausdorff and second countable. We shall always require that the manifolds underlying Lie groups be Hausdorff. Normally, we shall implicitly assume that they are second countable as well, but it is not essential as the following lemma shows.

Lemma 1.2. *Let G be a connected Lie group. Then G is second countable.*

Proof. Since G is a finite dimensional manifold its topology is first countable, meaning that every point p has a countable collection of open sets $V_n(p)$ that form a basis for the topology at that point. That is to say that any open subset containing the point p contains one of the $V_n(p)$. To show that a first countable space is second countable, one only needs show that it has a countable dense set. Since G is a finite dimensional manifold, there is a neighborhood U of the identity that is homeomorphic to an open ball in $\mathbb{R}^n$ for some finite n. Consequently, U has a countable dense set. Consider the subset W of G all of elements that can be written as finite products $g_1 \cdots g_k$ where each $g_i \in U$. Then W is clearly an open subset of G. We claim that W is also a closed subset of G. For suppose that g is a limit point of W. Choose a sequence, h_k, converging to g so that each $h_k \in W$. Then $h_k^{-1}g$ converges to $e \in G$ and hence for some (indeed all) k sufficiently large $h_k^{-1}g \in U$. Since, $h_k \in W$ and $h_k^{-1}g$ is in U, it follows that $g \in W$. Hence, W is both open and closed in G and is non-empty (since it contains $e \in G$). Since G is connected, $W = G$.

Now consider all finite products of a countable dense subset S of U. Since every element of G is a finite product of elements in U, a standard diagonalization argument shows that the set of elements represented by finite products of elements in S is dense in G. $\square$

Corollary 1.3. *A Lie group is second countable if and only if it has at most countably many connected components.*

Example. Let G be any Lie group; e.g., $(\mathbb{R}, +)$. Then the underlying group of G endowed with the discrete topology is a Lie group. If G is of positive dimension, this group is not second countable.

2 Examples of Lie Groups

Groups naturally arise as symmetry groups of some mathematical structure, so they come with their defining action. Most Lie groups, complex Lie groups, or linear algebraic groups arise in this way.

Any discrete group is a Lie group. If we require, as one often does, that a manifold must be second countable, they only the countable discrete groups are Lie groups. Of particular interest are the finite groups.

Example 1. Let Σ be the group of permutations of $\{1, \ldots, n\}$, i.e., bijections of this set onto itself. the product $\sigma\tau$ is defined to be the composition of the permutation σ followed by the permutation τ. The group axioms are straight-forward to verify. The group is of order $n!$.

Example 2 Let G be a group of order n. Then G can be embedded as a subgroup of Σ_n. Simply number the elements of G, i.e., set up a bijective correspondence between the elements of G and the set $\{1, \ldots, n\}$. Using this identification right multiplication by $g \in G$ becomes a permutation of $\{1, \ldots, n\}$ and the permutation associated to gh is the composition of the permutation associated to g followed by that associated to h. The identity element of G is the identity permutation, and inverses correspond. If right multiplication by $g \in G$ is the identity, then g is the identity in G. This proves that this correspondence embeds G as a subgroup of Σ_n.

Example 3. Symmetries of any mathematical structure naturally form a group under compositioin. For example, the symmetries of a square in the plane, meaning a Euclidean isometry of the square onto itself consists of rotations through multiples of $\pi/2$ around the central point of the square, together with flips, either about a line bisecting two opposite sides or a line passing through two opposite vertices. These form a group of order 8 with a normal subgroup being the group of 4 rotations. Similarly, the Euclidean symmetries of a regular n-gon in the plane is a group of order $2n$ with a normal subgroup being the n rotational symmetries. These groups are dihedral groups because the action of the quotient group of order two acts on the rotations by sending every rotation to its inverse.

Example 4. Let $P \subset \mathbb{R}^3$ be a regular solid. These are the regular tetrahedron, the cube, the octagon, the dodecahedron, and icosahedron. Then Euclidean symmetries of P is a finite subgroup. Problem: Compute the order of the symmetry group for each regular solid. Compute the order the the symmetry group for the n-cube.

Example 5. The Euclidean symmetries of the circle have a normal subgroup consisting of the rotations of the circle (a group isomorphic to the circle). The quotient is a group of order 2 that acts on S^1 by inverting the rotation. Show that the isometry group of the unit circle is $O(2)$, the group of 2×2 orthogonal matrices. More generally, the group of linear symmetries of the Euclidean distance on $\mathbb{R}^n$ is denoted $O(n) \subset GL(n, \mathbb{R})$. This group has two connected components, those of determinant 1 and those of determine -1. by $SO(n)$ we mean the component of the identity, namely the orientation-preserving subgroup of $O(n)$.

Example 6. The isometry group of the unit 2-sphere in $\mathbb{R}^3$ is the group of 3×3 orthogonal matrices, denoted $O(3)$. More generally, the isometry group of S^{n-1} is the group of $n] \times n$ orthogonal matrices $O(n)$. Compute the dimension of $O(n)$. Compute the tangent space to $O(n)$ at the identity inside the vector space of $n \times n$ matrices.

Example 7. Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{m}(z) > 0\}$. Then the group $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm 1\}$)with $SL(2, \mathbb{R})$ being the group of real two-by-two matrices over $\mathbb{R}$ of determinant 1) is the group of symmetries of the complex manifold $\mathbb{H}$. The action is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d}.$$

I have given a series of exercises to show that this defines an effective holomorphic action of $PSL(2, \mathbb{R})$ on $\mathbb{H}$ and that $PSL(2, \mathbb{R})$ is the full group of holomorphic symmetries of $\mathbb{H}$.

Example 8. A diffeomorphism of the unit 2-sphere is *conforma*l if at each point its differential is the product of an orientation-preseving orthogonal transformation of the tangent space and a scaling by a positive factor. Said another way a diffeomorphism is conformal if it is holomoprhic in the usual complex structure on S^2. The group of conformal transformations of the unit S^2 is the group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm 1\}$, where $SL(2, \mathbb{C})$ is the group of complex 2×2 matrices of determinant 1. The action is the projective linear action. We represent a point of S^2 as a pair $[z, \zeta]$ of complex numbers, not both zero, modulo the equivalence relation $[z, \zeta] = [\lambda z, \lambda \zeta]$ for any $\zeta \in \mathbb{C}^*$. The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts by sending $[z, \zeta]$ to $[az + b\zeta, cz + d\zeta]$. One often identifies the open set where $\zeta = 0$ with the complex plan with coordinate $w = z/\zeta$. Using this coordinate the action of the above matrix is

$$w \mapsto \frac{aw+b}{cw+d}.$$

One checks easily that this is an action by conformal transformations and only ± 1 act trivially. One again I have given exercises to show this is the full group of conformal symmetries of S^2.

Example 9. Consider the non-degenerate form

$$Q(x_0, \ldots, x_n) = -x_0^2 + \sum_{i=1}^{n} x_i^2$$

on $\mathbb{R}^{n+1}$. The locus $\{Q = 0\}$ is a double cone with singularity at the origin. The locus $\{Q = -1\}$ is a two-sheeted hyperboloid. The isometry group $O(Q)$ of linear transformations A of $\mathbb{R}^{n+1}$ that preserve Q in the sense that $Q(A(x)) = Q(x)$ for all $x \in \mathbb{R}^{n+1}$. is a group with four components. They are determined by (i) the determinant of A, which must be ± 1, and whether A preserves or reverses the two connected components of $\{Q = -1\}$. By $SO^+(Q)$ we mean the connected component of the identity in $O(Q)$, i.e., the symmetries of Q of determinant $+1$ that preserve the component $\{Q = -1\} \cap \{x_0 > 0\}$. of $\mathbb{Q} = -1\}$. We denote this sheet by $\mathbb{H}^n$ and call it *hyperbolic space* of dimension n. The restriction of the quadratic form Q to $T_p\mathbb{H}^n$ is positive definite, so that this restriction defines a Riemannian metric on $\mathbb{H}^n$. Clearly, $SO^+(Q)$ acts an orientation-preserving group of symmetries of this Riemannian manifold. It acts transitively, and the isotropy group of $(1, 0, \ldots, 0)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

where A is an arbitrary element in $SO(n)$. It follows by general principles that $SO^+(Q)$ is the orientation-preserving isometry group of $\mathbb{H}$ and that $\mathbb{H}$ has a metric of constant curvature. Direct computation shows the sectional curvatures are all -1.

In a formal sense, one can view $\mathbb{H}^n$ as the sphere of radius i in $\mathbb{R}^{n+1}$.

Example 10. There is a natural inclusion $SO(n-1) \subset SO(n)$ whose image fixes e_n. There is also a natural map $SO(n) \to S^{n-1}$ by sending A to Ae_n. The subgroup that fixes e_n is $SO(n-1)$ so that this action induces an isomorphism $SO(n)/SO(n-1) \to S^{n-1}$ and presents $SO(n)$ as a fibration over S^{n-1} with fibers being left cosets of $SO(n-1)$, i.e., the $gSO(n-1)$. In particular, we have a long exact sequence of homotopy groups

$$\cdots \to \pi_k(SO(n-1)) \to \pi_k(SO(n)) \to \pi_k(S^{n-1}) \to \pi_{k-1}(SO(n-1)) \to \cdots .$$

Since $\pi_1(S^{n-1}) = \pi_2(S^{n-1}) = 0$ for all $n \geq 4$, it follows that $\pi_1(SO(n)) = \pi_1(SO(3))$ for all $n \geq 3$. Furthermore $SO(3)$ is identified with pairs of unit vectors (x, y) with y orthogonal to x. The space of these pairs is identified with the unit circle tangent bundle of S^2 (x determines the point in S^2 and

y is a unit tangent vector at this point). This means we have a long exact sequence

$$\cdots \to \pi_2(S^2) \to \pi_1(SO(2)) \to \pi_1(SO(3)) \to 0.$$

The map $\pi_2(S^2) \to \pi_1(SO(2))$ is a map between groups identified with $\mathbb{Z}$, and hence is multiplication by some integer. That integer is the Euler characteristic of S^2, which is 2. hence $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, and the same isomorphism holds for $SO(n)$ for all $n \geq 3$.

Lemma 2.1. *Let G be a connected Lie group and let $\pi \colon \widetilde{G} \to G$ be a connected covering of G and fix $\widetilde{e} \in \pi^{-1}(e)$. Then there is a unique group structure on $\widetilde{G}$ with $\widetilde{e}$ as the identity element such that π is homomorphism. Also, give $\widetilde{G}$ the unique smooth structure so that π is a local diffeomorphism. This smooth structure and group structure make $\widetilde{G}$ a Lie group in such a way that π is a homomorphism of Lie groups. The kernel of π is contained in the center of $\widetilde{G}$.*

The proof is left as an exercise.

Example 11. It follows from the previous examples that for all $n \geq 3$ the Lie group $SO(n)$ has a universal covering that is a double cover. The double cove group of $SO(n)$ is defined to be $Spin(n)$. As we shall see, $Spin(n)$ most naturally constructed using Clifford Algebras.

3 Lie Algebras

3.1 The Basics

Definition 3.1. Fix a field K. A *Lie algebra* over K is a K-vector space V together with a bilienar map $V \otimes_K V \to V$ denoted by $X \otimes Y \mapsto [X, Y]$, called the *bracket* or the *Lie bracket* required to satisfy the following two axioms:

1. $[X, Y] = -[Y, X]$.
2. $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.

The second equation is called the *Jacobi Identity.* It can also be interpreted as saying that $[A, \cdot]$ is a derivation with respect to $[\cdot, \cdot]$, i.e.,

$$[A, [B, C]] = [[A, B], C] + [B, [A, C]].$$

Clearly, these algebraic equations make sense for vector spaces over any field K, though one often needs K to be of characteristic zero in many of the arguments. (Indeed, one can work with modules over a ring, defining what are called Lie rings, but this is beyond the scope of these lectures.) We are primarily (exclusively?) interested in the case of real and complex Lie algebras that are finite dimensional.

We will explain in the next lecture in more detail how Lie groups and Lie Algebras are related and where the Jacobi identity comes from, but for now we content ourselves with giving some examples.

Example 12. The space $M(n \times n, K)$ of $n \times n$ matrices with entries in K is a Lie algebra where the Lie bracket is given by $[A, B] = AB - BA$. Obviously, this bilinear map is skew-symmetric. To establish the Jacobi identity, we compute:

$$[A, [B, C]] = A(BC - CB) - (BC - CB)A$$
$$[C, [A, B]] = C(AB - BA) - (AB - BA)C$$
$$[B, [C, A]] = B(CA - AC) - (CA - AC)B.$$

Using the associativity of matrix multiplication we cancel these terms in pairs.

Example 13. Let $\mathcal{A}$ be an associative algebra over K. Then the computation in Example 12, is valid in $\mathcal{A}$ and shows that defining $[A, B] = AB - BA$ for all $A, B \in \mathcal{A}$ defines a Lie algebra structure on $\mathcal{A}$. This is the Lie algebra *determined* by the associative algebra. In fact, we shall show later in the course the Poincaré-Birkhoff-Witt Theorem which says that associated to a Lie algebra L there is an associative algebra $U(L)$ called the *universal enveloping algebra of* L. There is a injective linear map from $L \to U(L)$ compatible with the Lie bracket of L and the $AB - BA$ bracket on $U(L)$. This shows that the general Lie algebra L is a sub Lie algebra of the Lie algebra determined by an associate algebra. (The proof works over any field of characteristic 0.)

Example 14. Certain subspaces of $M(n \times n, \mathbb{R})$ are closed under the Lie bracket. For example, let $\mathfrak{o}(n)$ be the linear space of skew symmetric $n \times n$ real matrices,i.e., $\{X \mid X^{tr} = -X\}$ (where X^{tr} denotes the transpose of X). Then if X and Y are skew symmetric we have

$$(XY - YX)^{tr} = (Y^{tr}X^{tr} - X^{tr}Y^{tr}) = YX - XY,$$

showing that $[X, Y]$ is also skew symmetric.

Example 15. Another example is $n \times n$ matrices of trace zero. The point is that $Trace(XY) = Trace(YX)$, and hence $Trace([X,Y]) = 0$ for any pair of matrices X and Y.

Example 16. Let M be a smooth manifold and denote by $Vect(M)$ the vector space of smooth vector fields. The action of $Vect(M)$ on $C^\infty(M)$ identifies this space with the space of $\mathbb{R}$-linear maps $D\colon C^\infty(M) \to C^\infty(M)$ that are derivations in the sense that $D(fg) = D(f)g + fD(g)$. This space of first-order operators generates an associative algebra $\mathcal{D}(M)$ of differential operators on $C^\infty(M)$, with product being composition. The Lie bracket of vector fields is then induced from the $AB - BA$ bracket on $\mathcal{D}(M)$ making it a Lie algebra over $\mathbb{R}$. For vector fields X and Y, the composition XY is a second order operator (and hence is not a vector field);. Nevertheless, $XY - YX$ is a derivation (because the second-order terms cancel because of the equality of cross partial derivatives). Hence, $XY - YX$ is a vector field. This shows that the subspace of vector fields on M is a sub Lie algebra of the Lie algebra on $\mathcal{D}(M)$ defined from the associative multiplication on $\mathcal{D}(M)$. Indeed, $\mathcal{D}(M)$ is the universal enveloping algebra of the Lie algebra of vector fields.

Example 17.

Let G be a real Lie group and define

$$\mathrm{Ad}\colon G \times G \to G$$

by $(g, g') \mapsto gg'g^{-1}$. This is a smooth action of G (the first factor) on G (the second factor) preserving the identity element of the second factor. Differentiating at the identity of the second factor gives a linear representation $\mathrm{Ad}_G\colon G \times T_eG \to T_eG$. Differentiating this at the identity determines a map $\mathrm{ad}_G\colon T_eG \times T_eG \to T_eG$. In the next lecture, we shall prove that $\mathrm{ad}_G(X,Y)$ is a Lie bracket, giving T_eG the structure of a Lie algebra. It is called the *Lie algebra of* G and is denoted $\mathfrak{g}$. We shall show furthermore that in the case of $GL(n, \mathbb{R})$ the two definitions we have given of a Lie algebra structure on its tangent space are the same.

Example 18. The Lie algebra of $GL(n.\mathbb{R})$ as defined in Example 16 is $M(n \times n, \mathbb{R})$ with its Lie bracket being the bracket from Example 12. It is denoted $\mathfrak{g}l(n, \mathbb{R})$. It follows from this and the result stated in Example 17, that if $H \subset GL(n, \mathbb{R})$ is a Lie subgroup, then its tangent space at the identity, $\mathfrak{h} \subset \mathfrak{g}l(n, \mathbb{R})$ is closed under the $AB - BA$ bracket on $\mathfrak{g}l(n, \mathbb{R}) = M(n \times n, \mathbb{R})$ and the induced Lie algebra is the one given in Example 16 for the Lie group H.

3.2 Complex Lie algebras

A Lie algebra $(L, [\cdot,\cdot])$ is a *complex Lie algebra* if L is a complex vector space and if the pairing

$$[\cdot,\cdot]\colon L \times L \to L$$

is complex linear.

All of Examples 12 through 18 have comnplex analgoues:$M(n \times n, \mathbb{C})$ is a complex Lie algebra under the bracket $[A, B] = AB - BA$. An associative algebra over $\mathbb{C}$ determines a complex Lie algebra by the same formula. The Lie algebra of any complex Lie group is naturally a complex Lie algebra. The Lie algebra of $GL_n(\mathbb{C})$ is $M(n \times n, \mathbb{C})$. The:ie algebra of C^∞ vector fields on a complex manifold are a complex Lie algebra. The various subalgebras have complex analogous; e.g., $\mathfrak{o}(Q_{\mathbb{C}})$ where $Q_{\mathbb{C}}$ is a non-degenerate complex quadratic form on a finite dimensional complex vector space.