MORE ON SOLID ANALYTIC RINGS: DISCRETE HUBER PAIRS

1. Quasi-coherent sheaves on $\mathbb{P}^1$

We continue with further examples of solid rings that appear in algebraic and rigid geometry. To motivate their definition we will use the geometry of the projective space $\mathbb{P}^1$.

1.1. **Algebraic interpretation of ∞.** Classically, $\mathbb{P}^1$ can be constructed from two copies of the affine line $\mathbb{A}^1$, namely $\operatorname{Spec}\mathbb{Z}[T]$ and $\operatorname{Spec}\mathbb{Z}[T^{-1}]$, glued along $\mathbb{G}_m$, namely $\operatorname{Spec}\mathbb{Z}[T^{\pm 1}]$. Quasi-coherent sheaves on $\mathbb{P}^1$ are obtained by descent from quasi-coherent sheaves on $\operatorname{Spec}\mathbb{Z}[T]$ and $\operatorname{Spec}\mathbb{Z}[T^{-1}]$ that agree on $\mathbb{G}_m$. On the other hand, a quasi-coherent sheaf $\mathscr{F}$ on $\mathbb{P}^1$ has an excision sequence

$$0 \to \mathscr{F}[T^{-1}] \to \mathscr{F} \to \mathscr{F}|_{\operatorname{Spec}\mathbb{Z}[T]} \to H^1_{\infty}\mathscr{F} \to 0, \tag{1.1}$$

where $\mathscr{F}[T^{-1}]$ is the T^{-1}-torsion of $\mathscr{F}$, and $H^1_{\infty}\mathscr{F}$ is a higher cohomology group of sections of $\mathscr{F}$ supported at ∞. Geometrically, the sequence above arises from writing

$$|\mathbb{P}^1| = |\operatorname{Spec}\mathbb{Z}[T]| \bigsqcup |\operatorname{Spf}\mathbb{Z}[[T^{-1}]]| = |\mathbb{A}^1| \bigsqcup |\infty|,$$

where we understand the formal spectrum $\operatorname{Spf}\mathbb{Z}[[T^{-1}]]$ as the ind-scheme $\varinjlim_n \operatorname{Spec}\mathbb{Z}[T^{-1}]/(T^{-n})$. The (derived) categories of modules on $\operatorname{Spf}\mathbb{Z}[[T^{-1}]]$ can be then realised as both the category of (derived) T^{-1}-adically complete modules or the category of T^{-1}-torsion modules on $\mathbb{P}^1$.[1]

Thus, the sequence (1.1) is a consequence of the following localization sequence of ∞-derived categories

$$\mathscr{D}(\operatorname{Spf}\mathbb{Z}[[T^{-1}]]) \cong \mathscr{D}(\mathbb{P}^1)^{T^{-1}-\text{torsion}} \subset \mathscr{D}(\mathbb{P}^1) \xrightarrow{\otimes^L_{\mathbb{Z}[T]}} \mathscr{D}(\operatorname{Spec}\mathbb{Z}[T]).$$

Geometrically, the previous sequence describes a quasi-coherent sheaf on $\mathbb{P}^1$ as an extension of a sheaf on the affine space $\operatorname{Spec}\mathbb{Z}[T]$, and a sheaf supported at (the formal completion of) $\infty \in \mathbb{P}^1$.

1.2. **Solid interpretation of ∞.** Let us now consider the previous objects living in condensed mathematics. Let $\mathscr{D}(\mathbb{Z}_{\blacksquare})$ be the derived category of solid abelian groups. For any discrete ring A we can consider the induced analytic ring

$$(A,\mathbb{Z})_{\blacksquare} := (A, \operatorname{Mod}_A(\mathscr{D}(\mathbb{Z}_{\blacksquare}))),$$

and write $\mathscr{D}((A,\mathbb{Z})_{\blacksquare})$ for its derived category of complete modules. By definition, a condensed A-module is in $\mathscr{D}((A,\mathbb{Z})_{\blacksquare})$ if and only if its underlying condensed $\mathbb{Z}$-module structure is solid.

We can formally construct the category $\mathscr{D}(\mathbb{P}^1,\mathbb{Z}_{\blacksquare})$ of solid quasi-coherent sheaves of $\mathbb{P}^1$ by gluing the categories of modules $\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ and $\mathscr{D}((\mathbb{Z}[T^{-1}],\mathbb{Z})_{\blacksquare})$ along $\mathscr{D}((\mathbb{Z}[T^{\pm 1}],\mathbb{Z})_{\blacksquare})$.

Then, $\mathbb{Z}[[T^{-1}]]$ is now promoted from a discrete ring with some completeness property to a ring with an honest topology/condensed ring structure. Therefore, instead of taking T^{-1}-complete solid modules of $\mathbb{P}^1$ (i.e. the formal scheme $\operatorname{Spf}\mathbb{Z}[[T^{-1}]]$), we can consider the induced analytic ring

$$\mathbb{Z}[[T^{-1}]]_{\blacksquare} = (\mathbb{Z}[[T^{-1}]], \operatorname{Mod}_{\mathbb{Z}[[T^{-1}]]}(\mathscr{D}(\mathbb{Z}_{\blacksquare}))),$$

and write $\mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare})$ for its category of derived complete modules. In a previous lecture we saw that the free solid $\mathbb{Z}[[T^{-1}]]$-module generated by a profinite set $S = \varprojlim_i S_i$ was given by

$$\mathbb{Z}[[T^{-1}]]_{\blacksquare}[S] = \varprojlim_i \mathbb{Z}[[T^{-1}]][S_i],$$

which in turn is isomorphic to $\prod_I \mathbb{Z}[[T^{-1}]]$ for some index set I; these are compact projective generators of $\mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare})$.

By definition, $\mathbb{Z}[[T^{-1}]]_{\blacksquare}$ has the induced analytic structure from $\mathbb{Z}_{\blacksquare}$, namely, a condensed $\mathbb{Z}[[T^{-1}]]_{\blacksquare}$-module is complete if and only if its underlying condensed abelian group is $\mathbb{Z}_{\blacksquare}$-complete. Then, any solid

[1] We say that a quasi-coherent complex M in $\mathbb{P}^1$ is T^{-1}-torsion if $M \otimes^L_{\mathscr{O}_{\mathbb{P}^1}} \mathbb{Z}[T] = 0$, resp. T^{-1}-adically complete if $M = R\varprojlim_n (M \otimes^L_{\mathscr{O}(\mathbb{P}^1)} \mathbb{Z}[T^{-1}]/(T^{-n}).)$

T^{-1}-complete module over $\mathbb{Z}[T^{-1}]$ is a module over $\mathbb{Z}[[T^{-1}]]_{\blacksquare}$: this follows from stability under limits of complete modules on analytic rings. However, the category $\mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare})$ is larger! For instance, the algebra $\mathbb{Z}((T^{-1})) = \mathbb{Z}[[T^{-1}]][T]$ is a solid $\mathbb{Z}[[T^{-1}]]$-module which is not T^{-1}-adically complete.

We could then declare $\mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare})$ to be the category of sheaves of $\mathbb{P}^1_{\blacksquare}$ supported at ∞, and take $\mathbb{A}^1_{\blacksquare} = \mathbb{P}^1_{\blacksquare}\backslash\{\infty\}$ to be its complement. Concretely, we ask ourselves whether we have a localization sequence of categories

$$\mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare}) \to \mathscr{D}(\mathbb{P}^1) \to \mathscr{D}(\mathbb{A}^1_{\blacksquare}) \tag{1.2}$$

so that $\mathbb{Z}[T]_{\blacksquare} := (\mathbb{Z}[T], \mathscr{D}(\mathbb{A}^1_{\blacksquare}))$ defines a new analytic ring structure on the polynomial algebra $\mathbb{Z}[T]$. Furthermore, since $\mathscr{D}(\mathbb{P}^1)^{T^{-1}-\wedge} \subset \mathscr{D}(\mathbb{Z}[[T^{-1}]]_{\blacksquare})$, the fiber sequences (1.1) and (1.2) would imply that

$$\mathscr{D}(\mathbb{Z}[T]_{\blacksquare}) := \mathscr{D}(\mathbb{A}^1_{\blacksquare}) \subset \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}).$$

Therefore, by localizing (1.2) to $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[T]$ we should have a fiber sequence as follows:

$$\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare}) \to \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}) \to \mathscr{D}(\mathbb{Z}[T]_{\blacksquare}).$$

The realization of this idea lies in the following theorem:

Theorem 1.1 ([Sch19, Theorem 8.1])**.** *Consider the functor on condensed $\mathbb{Z}[T]$-modules mapping a profinite set $S = \varprojlim_i S_i$ to*

$$\mathbb{Z}[T]_{\blacksquare}[S] = \varprojlim_i \mathbb{Z}[T][S_i].$$

Then $\mathbb{Z}[T]_{\blacksquare}$ is an analytic ring over $\mathbb{Z}_{\blacksquare}$ with underlying ring $\mathbb{Z}[T]$. Moreover, we have a localization sequence

$$\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare}) \to \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}) \to \mathscr{D}(\mathbb{Z}[T]_{\blacksquare}).$$

More precisely, let $\iota : (\mathbb{Z}[T],\mathbb{Z})_{\blacksquare} \to (\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare}$ and $j : (\mathbb{Z}[T],\mathbb{Z})_{\blacksquare} \to \mathbb{Z}[T]_{\blacksquare}$ be the natural morphisms of analytic rings. The following holds:

1. *The $(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}$-algebra $\mathbb{Z}((T^{-1}))$ is compact and idempotent. We let $\iota_* : \mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare}) \to \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ denote the forgetful functor, and let ι^* and $\iota^!$ be its left and right adjoint respectively, namely*
$$i^*M = \mathbb{Z}((T^{-1})) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}} M$$
and
$$\iota^! M = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})), M).$$
2. *The base change functor $j^* : \mathbb{Z}[T]_{\blacksquare} \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}} - : \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}) \to \mathscr{D}(\mathbb{Z}[T]_{\blacksquare})$ has a fully faithful right adjoint j_* (the forgetful functor) such that*
$$j_*j^*M = R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}[\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))], M).$$
In particular, we have a natural equivalence of categories
$$\mathscr{D}(\mathbb{Z}[T]_{\blacksquare}) = \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})/\mathscr{D}(((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare})).$$
3. *The base change functor j^* has a fully faithful left adjoint $j_!$ given by*
$$j_!j^*M = (\mathrm{fib}[\mathbb{Z}[T] \to \mathbb{Z}((T^{-1}))]) \otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}} M.$$

Furthermore, we have excision fibrations for $M \in \mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$

$$j_!j^*M \to M \to \iota_*\iota^*M$$

and

$$\iota_*\iota^!M \to M \to j_*j^*M.$$

Proof. Most of the proposition will follow from the properties of $\mathbb{Z}((T^{-1}))$ (i.e. compact and idempotent) thanks to the localization sequence that one obtains at the level of categories, see [CS22, Construction 5.2]. Let us explain the main steps in the proof:

Step 1. First let us show that $\mathbb{Z}((T^{-1}))$ is a compact $(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}$-algebra. This follows from the resolution

$$0 \to \mathbb{Z}[[X]] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \xrightarrow{XT-1} \mathbb{Z}[[X]] \otimes_{\mathbb{Z}} \mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \to 0 \tag{1.3}$$

and the fact that $(\prod_I \mathbb{Z})\otimes_{\mathbb{Z}} A$ is a family of compact projective generators for $\mathscr{D}((A,\mathbb{Z})_{\blacksquare})$ and any discrete ring A.

Step 2. The $(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}$-algebra $\mathbb{Z}((T^{-1}))$ is idempotent. This follows from the exact sequence (1.3), namely, one gets that

$$\begin{aligned}\mathbb{Z}((T^{-1}))\otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}}\mathbb{Z}((T^{-1})) &= \mathrm{cofib}(\mathbb{Z}[[X]]\otimes_{\mathbb{Z}_{\blacksquare}}\mathbb{Z}((T^{-1}))\xrightarrow{XT-1}\mathbb{Z}[[X]]\otimes_{\mathbb{Z}_{\blacksquare}}\mathbb{Z}((T^{-1})))\\ &= \mathrm{cofib}(\mathbb{Z}[[X,T^{-1}]][T]\xrightarrow{X-T^{-1}}\mathbb{Z}[[X,T^{-1}]][T])\\ &= \mathbb{Z}((T^{-1})).\end{aligned}$$

Formally we deduce that $\iota_*:\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare})\subset\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ is a full subcategory stable under limits and colimits, with inclusion having left and right adjoint

$$\iota^*M=\mathbb{Z}((T^{-1}))\otimes^L_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}}M \text{ and } \iota^!M=R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),M).$$

In particular, $\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare})$ defines an analytic ring structure on $\mathbb{Z}[T]!$.

Step 3. Construction of $\mathscr{D}(\mathbb{Z}[T]_{\blacksquare})$. By step 2, $\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare})$ is a thick tensor-ideal of $\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ stable under all limits and colimits. We can then define the quotient category

$$\mathscr{C}:=\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})/\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_{\blacksquare}).$$

We have a localization functor $j^*:\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})\to\mathscr{C}$, and j^* has fully faithful left and right adjoints satisfying

$$j_!j^*M=\mathrm{fib}(\mathbb{Z}[T]\to\mathbb{Z}((T^{-1})))\otimes_{(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}}M \text{ and } j_*j^*M=R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}(\mathbb{Z}[T]\to\mathbb{Z}((T^{-1}))),M)$$

for $M\in\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$. Moreover, we have excision fiber sequences

$$j_!j^*M\to M\to\iota_*\iota^*M$$

and

$$\iota_*\iota^!M\to M\to j_*j^*M.$$

See [CS22, Lecture V] for more details. Our next task is to show that the fully faithful functor $j_*:\mathscr{C}\to\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ defines the analytic ring $\mathbb{Z}[T]_{\blacksquare}$.

Step 4. We need to prove that $j_*\mathscr{C}\subset\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ is stable under limits, colimits and mapping spaces from profinite sets (i.e. tensored over $\mathscr{D}(\mathrm{CondAb})$). Stability under limits follows formally since j_* is a right adjoint. To see stability under colimits, note that

$$j_*j^*M=R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathrm{fib}(\mathbb{Z}[T]\to\mathbb{Z}((T^{-1}))),M),$$

and that $\mathrm{fib}(\mathbb{Z}[T]\to\mathbb{Z}((T^{-1})))$ is a compact $(\mathbb{Z}[T],\mathbb{Z})_{\blacksquare}$-module. This implies that j_* commutes with colimits as wanted. Finally, the same explicit description of j_*j^*M shows that for any profinite set S we have

$$j_*j^*R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S],M)=R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[S],j_*j^*M).$$

Since $j_*\mathscr{C}\subset\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_{\blacksquare})$ is the full subcategory of objects M such that $M\to j_*j^*M$ is an equivalence, the pair $(\mathbb{Z}[T],j_*\mathscr{C})$ defines an analytic ring structure on $\mathbb{Z}[T]$ by [CS20, Proposition 12.20].

Step 5. Finally, we need to compute, for S a profinite set, the compact projective generator of $j_*\mathscr{C}$ generated by S, namely,

$$j_*j^*(\mathbb{Z}_{\blacksquare}[S]\otimes_{\mathbb{Z}}\mathbb{Z}[T]).$$

We first prove the case of $S=*$, we have an excision sequence

$$\iota_*\iota^!\mathbb{Z}[T]\to\mathbb{Z}[T]\to j_*j^*\mathbb{Z}[T].$$

We need to prove that

$$\iota_*\iota^!\mathbb{Z}[T]=R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1})),\mathbb{Z}[T])=0.$$

By step 1 we have that

$$\begin{aligned}\iota_*\iota^!\mathbb{Z}[T] &= \mathrm{fib}(R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[X]]\otimes_{\mathbb{Z}}\mathbb{Z}[T],\mathbb{Z}[T]) \xrightarrow{XT-1} R\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[[X]]\otimes_{\mathbb{Z}}\mathbb{Z}[T],\mathbb{Z}[T]))\\ &= \mathrm{fib}(R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[X]]),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T] \xrightarrow{XT-1} R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[X]]),\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[T])\\ &= \mathrm{fib}(\mathbb{Z}[X^{\pm 1}]/X\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[T] \xrightarrow{XT-1} \mathbb{Z}[X^{\pm 1}]/X\mathbb{Z}[X]\otimes_{\mathbb{Z}}\mathbb{Z}[T])\\ &= 0,\end{aligned}$$

obtaining what we wanted.

We now prove the claim for general S. Recall that by construction, the kernel of j^* is precisely $\mathscr{D}((\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare)$. Then, we need to show that the quotient

$$Q(S) := \mathrm{cofib}(\mathbb{Z}_\blacksquare[S]\otimes_{\mathbb{Z},\blacksquare}\mathbb{Z}[T]\to\mathbb{Z}[T]_\blacksquare[S]) \tag{1.4}$$

has a natural structure of $\mathbb{Z}((T^{-1}))$-module. Indeed, if this holds true, we get that

$$j_*j^*(\mathbb{Z}_\blacksquare[S]\otimes_{\mathbb{Z}}\mathbb{Z}[T]) = j_*j^*\mathbb{Z}[T]_\blacksquare[S],$$

but $\mathbb{Z}[T]_\blacksquare[S] = \varprojlim_i \mathbb{Z}[T][S_i]$ is a limit of finite free $\mathbb{Z}[T]$-modules, so that $j_*j^*\mathbb{Z}[T]_\blacksquare[S] = \mathbb{Z}[T]_\blacksquare[S]$ since $j_*\mathscr{C}$ is stable under limits by step 4.

Step 6. In this final step we show that (1.4) has a natural structure of $\mathbb{Z}((T^{-1}))$-module. Let us fix an isomorphism $\mathbb{Z}_\blacksquare[S]=\prod_I\mathbb{Z}$, it suffices to see that there is an equivalence of $\mathbb{Z}[T]$-modules

$$Q(S) = Q'(S) := \mathrm{cofib}((\prod_I\mathbb{Z}[[T^{-1}]])[T]\to\prod_I\mathbb{Z}((T^{-1}))).$$

Consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_I\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}[T] & \longrightarrow & \prod_I\mathbb{Z}[T] & \longrightarrow & Q(S) & \longrightarrow & 0\\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & \prod_I\mathbb{Z}[[T^{-1}]][T] & \longrightarrow & \prod_I\mathbb{Z}((T^{-1})) & \longrightarrow & Q'(S) & \longrightarrow & 0.\end{array}$$

By the snake lemma we have a long exact sequence

$$0\to\ker f\to\ker g\to\ker h\to\mathrm{coker}\, f\to\mathrm{coker}\, g\to\mathrm{coker}\, h\to 0.$$

It is clear that $\ker f=\ker g=0$, to show that h is an isomorphism we just need that $\mathrm{coker}\, f\cong\mathrm{coker}\, g$ is an isomorphism, but both terms can be identified with $T^{-1}\prod_I\mathbb{Z}[[T^{-1}]]$, proving what we wanted. Equivalently, the square

$$\begin{array}{ccc} \prod_I\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}[T] & \longrightarrow & \prod_I\mathbb{Z}[T]\\ \downarrow & & \downarrow\\ \prod_I\mathbb{Z}[[T^{-1}]][T] & \longrightarrow & \prod_I\mathbb{Z}((T^{-1}))\end{array}$$

is a pushout square in $\mathscr{D}((\mathbb{Z}[T],\mathbb{Z})_\blacksquare)$, which directly implies that $Q(S)\cong Q'(S)$ as $\mathbb{Z}[T]$-modules. ☐

Remark 1.2. Note that the definition of $\mathbb{Z}[T]_\blacksquare$ via a functor of measures is analogue to that of $\mathbb{Z}_\blacksquare$. In Proposition 2.1 we will construct even more examples of solid rings using this idea.

Remark 1.3. We now explain a more clear relation of the previous construction with rigid geometry. Let p be a prime number, and let $\mathbb{Z}_{p,\blacksquare}$ be the induced analytic ring structure from $\mathbb{Z}$ to $\mathbb{Z}_p$. The compact projective generators of $\mathscr{D}(\mathbb{Z}_{p,\blacksquare})$ are the modules of the form

$$\mathbb{Z}_p\otimes^L_{\mathbb{Z}_\blacksquare}\prod_I\mathbb{Z}=\prod_I\mathbb{Z}_p.$$

We let $\mathbb{Q}_{p,\blacksquare}$ also denote the induced analytic ring structure from $\mathbb{Z}$ to $\mathbb{Q}_p$, the compact projective generators of $\mathscr{D}(\mathbb{Q}_{p,\blacksquare})$ have the form $(\prod_I\mathbb{Z}_p)[\frac{1}{p}]$.

In Theorem 1.1 we constructed an analytic ring $\mathbb{Z}[T]_\blacksquare$ over the polynomial algebra, such that the objects $\prod_I\mathbb{Z}[T]$ are compact projective generators in $\mathscr{D}(\mathbb{Z}[T]_\blacksquare)$. This analytic ring structure was constructed as the complement of the analytic ring $(\mathbb{Z}[[T^{-1}]],\mathbb{Z})_\blacksquare$ over $\mathbb{P}^1_{\mathbb{Z}}$. Therefore, both $\mathbb{Z}[T]_\blacksquare$ and $(\mathbb{Z}((T^{-1})),\mathbb{Z})_\blacksquare$

define subspaces of $\mathbb{P}^1_{\mathbb{Z}}$ corresponding to $\mathbb{A}^1$ and ∞ with respect to its Zariski spectrum. A natural question is to describe the behavior of the pullback of these subspaces to other condensed subrings, for example $\mathbb{Z}_p$ or $\mathbb{Q}_p$. It turns out that these new spaces are very well explained using Huber's theory of adic spaces.

The algebra $\mathbb{Q}_p \otimes^L_{\mathbb{Z},\blacksquare} \mathbb{Z}[[T^{-1}]]$ is equivalent to $\mathbb{Z}_p[[T^{-1}]][\frac{1}{p}]$, which consists on the bounded functions of an open disc of radius 1 around $\infty \in \mathbb{P}^1_{\mathbb{Q}_p}$, namely, $\overline{\mathbb{D}}_{\mathbb{Q}_p}(\infty, 1)$. The complement of $\overline{\mathbb{D}}_{\mathbb{Q}_p}(\infty,1)$ in $\mathbb{P}^1_{\mathbb{Q}_p}$ should be then the open affinoid disc of radius 1 around 0, namely $\mathbb{D}_{\mathbb{Q}_p}(0,1) = \mathrm{Spa}(\mathbb{Q}_p\langle T\rangle, \mathbb{Z}_p\langle T\rangle)$. It turns out that the base change of analytic rings $\mathbb{Q}_p \otimes^L_{\mathbb{Z},\blacksquare} \mathbb{Z}[T]_\blacksquare$ is given by the Tate algebra $\mathbb{Q}_p\langle T\rangle_\blacksquare$, with a family of compact projective generators

$$(\prod_I \mathbb{Z}_p\langle T\rangle)[\frac{1}{p}].$$

We see then that the datum defining $\mathbb{Q}_p\langle T\rangle_\blacksquare$ consists on the analytic ring $\mathbb{Q}_p\langle T\rangle$ (the first factor on the Huber pair defining $\mathbb{D}_{\mathbb{Q}_p}(0,1)$), and on the open bounded subring $\mathbb{Z}_p\langle T\rangle$ (the second factor of the Huber pair). A more explicit relation between (discrete) Huber rings and analytic rings will be discussed in §2.

2. Discrete Huber pairs

The construction of the analytic ring $\mathbb{Z}[T]_\blacksquare$ of Theorem 1.1 can be slightly generalized.

Proposition 2.1. *Let A be an algebra of finite type over $\mathbb{Z}$. For a profinite set $S = \varprojlim_i S_i$ consider*

$$A_\blacksquare[S] = \varprojlim_i A[S_i].$$

Then A defines an analytic ring structure over A.

Proof. By an inductive argument, one can handle the case of a polynomial algebra $A = \mathbb{Z}[T_1, \ldots, T_n]$, namely, in the proof of Theorem 1.1 we just needed the Steps 1-6 to hold, and this would work over any base $A_\blacksquare$ (once $A_\blacksquare$ is an analytic ring). Now, let A be a quotient of $\mathbb{Z}[T_1, \ldots, T_n]$, we claim that the induced analytic ring $(A, \mathrm{Mod}_A(\mathscr{D}(\mathbb{Z}[T_1, \ldots, T_n]_\blacksquare)))$ defines $A_\blacksquare$. It suffices to prove that

$$\prod_I \mathbb{Z}[T_1, \ldots, T_n] \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]} A = \prod_I A.$$

The ring $\mathbb{Z}[T_1, \ldots, T_n]$ is noetherian, since A is a finitely generated module we can find a resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to A \to 0$$

where each P_i a finite free $\mathbb{Z}[T_1, \ldots, T_n]$-module. Thus, we get that

$$\begin{aligned}\prod_I \mathbb{Z}[T_1, \ldots, T_n] \otimes^L_{\mathbb{Z}[T_1,\ldots,T_n]} A &= [\cdots \prod_I P_2 \to \prod_I P_1 \to \prod_I P_0] \\ &= \prod_I [\cdots \to P_2 \to P_1 \to P_0] \\ &= \prod_I A\end{aligned}$$

as wanted. □

Even more generally, we can combine induced analytic ring structures together with the analytic rings of Proposition 2.1:

Definition 2.2. Let (A, S) be a pair consisting on a discrete ring A and a set of elements $S \subset A$. We define the analytic ring $(A, S)_\blacksquare$ by taking the underlying condensed ring A, and declaring an A-module M to be $(A, S)_\blacksquare$-complete if for all $s \in S$ the restriction of M to $\mathbb{Z}[s]$-module is $\mathbb{Z}[s]_\blacksquare$-complete as in Theorem 1.1.

Different sets $S \subset A$ provide different analytic structures on A, however, the map $(A, S) \mapsto (A, S)_\blacksquare$ is not an injection. One could ask what is the maximal set $S' \subset A$ containing S such that $(A, S)_\blacksquare = (A, S')_\blacksquare$, it turns out that this is naturally explained using Huber's theory of affinoid rings:

Proposition 2.3. *Let (A, S) be as in Definition 2.2. Then there is a maximal set $S \subset S'$ such that $(A, S)_\blacksquare = (A, S')_\blacksquare$ satisfying the following properties:*

(1) *The set S' is the ring given by the integral closure of $\mathbb{Z}[S]$ in A.*
(2) *If A is of finite type over $\mathbb{Z}$, then we have that $(A, A)_\blacksquare = A_\blacksquare$ as in Proposition 2.1.*

Proof. Part (2) follows from part (1) and Proposition 2.1 after taking a surjection from a polynomial algebra. Note also that for any $a \in A$ we have a map of analytic rings $\mathbb{Z}[a]_\blacksquare \to A_\blacksquare$, namely, the product $\prod_I A$ is already $\mathbb{Z}[a]_\blacksquare$-complete being a product of discrete $\mathbb{Z}[a]$-modules.

We now prove part (1). First, we describe the compact projective generators of $(A, S)_\blacksquare$. First, let us write $S = \bigcup_i S_i$ as an union of finite sets, then we have that

$$(A, S)_\blacksquare = \varinjlim_i (A, S_i)_\blacksquare$$

(namely, $\mathscr{D}((A, S)_\blacksquare) = \bigcap_i \mathscr{D}((A, S_i)_\blacksquare)$ as full subcategories of $\mathscr{D}((A, \mathbb{Z})_\blacksquare)$). For each S_i let $B_i \subset A$ be the finitely generated subring generated by S_i, then the same proof of Proposition 2.1 shows that

$$(A, S_i)_\blacksquare[K] = A \otimes_{B_i} B_{i,\blacksquare}[K]$$

for K a profinite set (take the polynomial algebra generated by S_i and the induced analytic structure on B_i and A). Let $B = \varinjlim_i B_i$, one deduces that

$$(A, S_i)_\blacksquare[K] = A \otimes_B B_\blacksquare[K]$$

with $B_\blacksquare[K] = \varinjlim_i B_{i,\blacksquare}[K]$.

Let us fix an isomorphism $\mathbb{Z}_\blacksquare[K] = \prod_I \mathbb{Z}$. Let A^+ be the integral closure of B in A and let $a \in A^+$, then there is a polynomial $p(T) = T^n + b_{n-1}T^{n-1} + \cdots + b_0$ over some B_i such that $P(a) = 0$. Consider the polynomial algebra $B_i[T]$, it suffices to show that the map $T \mapsto a$ extend to a morphism of analytic rings

$$(B_i[T]/p(T))_\blacksquare \to (A, S)_\blacksquare,$$

but $B_i[T]/p(T)$ is a finite free B_i-module, so that

$$\prod_I B_i[T]/p(T) = (B_i[T]/p(T)) \otimes_{B_i} \prod_I B_i,$$

and $(B_i[T]/p(T))_\blacksquare$ has the induced analytic structure from $B_{i,\blacksquare}$. This provides morphisms of analytic rings

$$\mathbb{Z}[T]_\blacksquare \to B_i[T]_\blacksquare \to (B_i[T]/p(T))_\blacksquare = (B[T]/p(T), B_i)_\blacksquare \to (A, S)_\blacksquare$$

as wanted.

Conversely, let $a \in A$ be such that the map $(\mathbb{Z}[a], \mathbb{Z})_\blacksquare \to (A, S)_\blacksquare$ factors through $\mathbb{Z}[a]_\blacksquare \to (A, S)_\blacksquare$. For each B_i let $B'_i = B_i[a]$ be the subring of A generated by B_i and a, and let $B' = \varinjlim B'_i$. Then the assumption on a shows that for any K profinite

$$(A, S)_\blacksquare[K] = A \otimes_B (\varinjlim_i \prod_I B_i) = A \otimes_{B'} (\varinjlim_i \prod_I B'_i).$$

An equivalent way to write down the previous colimits is as follows:

$$(A, S)_\blacksquare[K] = \varinjlim_{M, B_i} \prod_I M$$

where $M \subset A$ runs over all the finitely generated submodules of B_i in A. Taking $I = \mathbb{N}$, the sequence $(a, a^2, \ldots) \in \prod_\mathbb{N} B'_i \subset \varinjlim_{M,B_i} \prod_\mathbb{N} M$. Therefore, there is some j and some finitely generated B_j-module $M \subset A$ such that $(a, a^2, a^3, \ldots) \in \prod_\mathbb{N} M$. This proves that the algebra $B'_j = B_j[a]$ is a submodule of M, and so that a is integral over B_j. This proves that the maximal set S' in (1) is the integral closure of B in A, showing the proposition. □

We naturally arrive to the notion of a discrete Huber pair:

Definition 2.4. A discrete Huber pair is the datum (A, A^+) of a discrete ring A and an integrally closed subring $A^+ \subset A$. A morphism of discrete Huber pairs $(A, A^+) \to (B, B^+)$ is a morphism of discrete rings $A \to B$ mapping A^+ to B^+.

Corollary 2.5. *There is a fully faithful embedding from the category of discrete Huber pairs into the category of analytic rings over $\mathbb{Z}_\blacksquare$ given by*

$$(A, A^+) \mapsto (A, A^+)_\blacksquare.$$

Proof. By Proposition 2.3 we can recover the ring A^+ as the set of elements $a \in A$ such that the map $(\mathbb{Z}[T], \mathbb{Z}) \to (A, A^+)_{\blacksquare}$ of analytic rings factors through $(\mathbb{Z}[T], \mathbb{Z}) \to \mathbb{Z}[T]_{\blacksquare} \to (A, A^+)_{\blacksquare}$. This shows the conservativity of $(A, A^+) \mapsto (A, A^+)_{\blacksquare}$. Let us now take (A, A^+) and (B, B^+) be two Huber rings, and consider a map of analytic rings $(A, A^+)_{\blacksquare} \to (B, B^+)_{\blacksquare}$. By definition the space $\mathrm{Map}_{\mathrm{AnRing}}((A, A^+)_{\blacksquare}, (B, B^+)_{\blacksquare})$ is the full subspace of the mapping space of condensed rings $\mathrm{Map}_{\mathrm{CondRing}}(A, B)$ such that the restriction of $M \in \mathscr{D}((B, B^+)_{\blacksquare})$ to an A-module is $(A, A^+)_{\blacksquare}$-complete. But Proposition 2.3 implies that for $a \in A^+$ we have a composition of analytic rings

$$\mathbb{Z}[a]_{\blacksquare} \to (A, A^+)_{\blacksquare} \to (B, B^+)_{\blacksquare},$$

and so the image of a in B must land in B^+ by the same proposition. This finishes the proof. □

The theory of (complete) Huber pairs and adic spaces can be better explained and generalized using condensed mathematics. For a better reference towards this direction we recommend Lectures VII-X of the course in Analytic Stacks held by Clausen and Scholze: `https://people.mpim-bonn.mpg.de/scholze/AnalyticStacks.html`, see also [And21].

References

[And21] Grigory Andreychev. Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces. `https://arxiv.org/abs/2105.12591`, 2021.

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`, 2020.

[CS22] Dustin Clausen and Peter Scholze. Condensed Mathematics and Complex Geometry. `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`, 2022.

[Sch19] Peter Scholze. Lectures on Condensed Mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.