Basics of Condensed Mathematics

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1 The Definitions

Let **Prof** denote the category whose objects are profinite sets, or equivalently, compact, totally disconnected Hausdorff spaces, and whose morphsims are conitinuous maps.

Definition 1.1. A finite collection of morphisms $\{S_i \to S\}_i$ of **Prof** with a given codomain is a *covering* if $\coprod_i S_i \to S$ is a surjective map of topological spaces (or equivalently of sets).

The category **Prof** has all limits, finite colimits, and fibered products so that if $T \to S$ is a morphism and $\{S_i \to S\}_i$ is a covering we can form $\{S_i \times_S T \to T\}_i$. It is easily seen to be a covering. Also clearly if $\{S_i \to S\}_i$ is a covering and, for each i, there is a covering $\{S_{i,j} \to S_i\}_j$, then $\{S_{i,j} \to S\}_{i,j}$ is a covering. That is to say this notion of coverings for **Prof** determines a Grothendieck site.

Definition 1.2. A *condensed set* is a sheaf of sets for this Grothendieck site. This means that there is a congravariant functor $\mathcal{T} \colon \mathbf{Prof}^{\mathrm{op}} \to Sets$ so that for any covering $\{U_j \to U\}_j$ the following diagram expresses the first term as the equalizer of the two maps from the second term to he third:

$$\mathcal{T}(S) \xrightarrow{\alpha^*} \prod_j \mathcal{T}(S_j) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{j,j'} \mathcal{T}(S_j \times_S S_{j'})$$

Replacing the target category, sets, by the category of abelian groups produces the category of condensed abelian groups.

In more down to earth language, a condensed set is a contravariant functor $\mathcal{T} \colon \mathbf{Prof} \to Sets$ such that:

1. for each surjection $\alpha\colon S' \to S$ in **Prof** the diagram

$$\mathcal{T}(S) \xrightarrow{\alpha^*} \mathcal{T}(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{T}(S' \times_S S')$$

   expresses $\mathcal{T}(S)$ as the equalizer of p_1^* and p_2^*, and

2. for any finite coprod $\coprod_{i\in I} S_i$ we have an isomorphism

$$\mathcal{T}(\coprod_{i\in I} S_i) = \prod_{i\in I} \mathcal{T}(S_i),$$

   given by the product of the maps induced by the inclusions $S_i \to \coprod_{i\in I} S_i$.

A morphism $\varphi\colon \mathcal{T} \to \mathcal{T}'$ is a family of set functions $\varphi_S\colon \mathcal{T}(S) \to \mathcal{T}'(S)$ as S ranges over the profinite sets that are compatible under pullbacks $S' \to S$.

As is usual with sheaves, these are the objects of a category, the *category of condensed sets* with morphisms being morphisms of sheaves on the Grothendieck site. A *condensed abelian group* is a sheaf of abelian groups for this Grothendieck site. These are the objects of a subcategory of the category of condensed sets with morphisms being morphisms of sheaves of groups. Condensed abelian groups are exactly the abelian group objects in the category of condensed sets.

Definition 1.3. For any condensed set $\mathcal{T}$ the value $\mathcal{T}(*)$, where $*$ represents the one-point space is called the *underlying set* of the condensed set.

Remark 1.4. Let S be a profinite set. We define a condensed set, denoted $\underline{S}$ by associating to the profinite set S' the set $\mathrm{Hom}_{cont.}(S', S)$. It is a straightforward exercise to show that this determines a sheaf for the Grothendieck site. These are called *representable* condensed sets. Clearly $\underline{S}(*) = S$.

Claim 1.5. *Let $\underline{S}$ be a representable condensed set and T an arbitrary condensed set. Then*

$$\mathrm{Hom}(\underline{S}, T) = T(S).$$

Proof. This is a Yoneda limit argument: Given $\alpha \in \mathrm{Hom}(\underline{S}, T)$ we associate $\alpha(\mathrm{Id}_S) \in T(S)$. Conversely, given an element $a \in T(S)$ for each $\rho\colon S' \to S$ we have $\rho^* a \in T(S')$. These are compatible under pullback and determine an element of T. These are easily seen to be inverse bijections. $\square$

Corollary 1.6. $Hom(\underline{S}, \underline{T}) = Hom_{cont}(S, T)$.

Proof. $Hom(\underline{S}, \underline{T}) = \underline{T}(S) = T(S)$. $\square$

There is a generalization of $S \mapsto \underline{S}$. For any topological space X we can define a sheaf $\underline{X}$ on **Prof** by defining $\underline{X}(S) = Hom(S, X)$. While these are reasonable condensed sets (for reasonable X), they are not representable condensed sets unless X is profinite.

2 Basic Lemmas

2.1 Relations holding in Prof that hold in Condensed sets

Lemma 2.1. *There are indexed products and fibered products in the category of κ-condensed sets.*

Proof. Given condensed sets $\{T_i\}_{i\in I}$, the presheaf $S \mapsto \prod_i T_i(S)$ is easily seen to be a sheaf. Likewise given T_1, T_2 mapping to T the presheaf $S \mapsto T_1(S) \times_{T(S)} \times T_2(S)$ is easily seen to be a sheaf. $\square$

We turn now to co-products in the category of condensed sets. Suppose that $\{T_i\}_{i\in I}$ is an indexed family of condensed sets. Then the contravariant functor $S \mapsto \coprod_{\in I} T_i(S)$ is a presheaf on **Prof**. But if the cardinality of I is greater than one, it is not a sheaf since

$$\coprod_{i\in I} T_i\left(S_1 \coprod \cdots \coprod S_k\right) = \coprod_{i\in I}\left(\prod_{j=1}^{k} T_i(S_j)\right) \neq \prod_{j=1}^{k}\left(\coprod_{i\in I} T_i(S_j)\right).$$

We denote by $\coprod_i X_i$ the sheafification of the above presheaf. Using the universal property of sheafification and the fact that the co-product in the category of presheaves is a categorical co-product, it is straightforward to show that this sheaf satisfies the categorical co-product axiom in the category of sheaves on **Prof**.

Since the T_i are sheaves, for any covering $\{S_j \to S\}$ the pullbback

$$\coprod_i T_i(S) \to \prod_j\left(\coprod_i T_i(S_j)\right)$$

is an injection and hence the value of the sheaf $\coprod_i T_i(S)$ is the coilmit over the indexed set of coverings $\{S_j \to S\}$ of the equalizer of $\prod_j(\coprod_i T_i(S_j)$ under the pullback under the two projection mappings to $\prod_{j,j'}\left(\coprod_i T_i(S_j \times_S S_{j'})\right)$.

Lemma 2.2. *Let $T = T_1 \coprod \cdots \coprod T_n$ be decomposition of a profinite set T into a finite disjoint union of open and closed subsets. Then in the category of condensed sets there is a natural identification of condensed sets $\underline{T} = \underline{T_1} \coprod \cdots \coprod \underline{T_n}$.*

Proof. Let U be the presheaf of **Prof** defined by

$$S \mapsto U(S) = \underline{T_1}(S) \coprod \cdots \coprod \underline{T_n}(S).$$

Clearly, for every profinite set S, there is an embedding $U(S) \subset \underline{T}(S)$, making U a sub-presheaf of $\underline{T}$. Thus, the sheafification of U is a subsheaf of $\underline{T}$. Recall that in this case the sheafification of U is defined by sending S to the colimit over all covers $\{S'_j \to S\}_j$ of the equalizer

$$\prod_j U(S'_j) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{j,k} U(S'_j \times_S S'_k)$$

Thus, to prove that the sheafification of U is equal to $\underline{T}$ we need only show that for every profinite set S and every $\alpha \in \underline{T}(S) = \mathrm{Hom}(S,T)$ there is a covering $\{S'_i \to S\}_i$ and an element a of the equalizer of

$$\prod_j U(S'_j) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{j,k} U(S'_j \times_S S'_k)$$

such that a maps to α, under the natural identication of $\underline{T}(S)$ with the equlalizer of

$$\prod_i \underline{T}(S'_j) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{j,k} \underline{T}(S'_j \times_S S'_k).$$

and the inclusion from the first equalizer to the second. For then the element a represents an element in the colimit that is the definition of the value of the sheafification of U on S and under the natural inclusion of the sheafification of U into $\underline{T}$, this element maps to $\alpha \in \underline{T}(S)$.

We take a decomosition $S = \coprod S_j$ such that under $\alpha \in \underline{T}(S) = \mathrm{Hom}(S,T)$ each S_j maps to one of the components $T_{i(j)}$. Then define the covering to be $\{S_i \subset S\}_i$. The equalizer in question is simply $\prod_i U(S_j) = \prod_j (\coprod_i \underline{T}_i(S_j))$. Since for each j there is $i(j)$ such that $\alpha(S_j) \subset T_{i(j)}$, it follows immediately that α is contained in $\prod_j U(S_j) \subset \prod_j T(S_j)$. $\square$

Since profinite sets are compact, the same argument works for all indexed co-products

$$\left(\coprod_i \underline{T_i}\right)_{sh} = \underline{\coprod_i T_i},$$

but of course these are not representable sheaves.

Lemma 2.3. *For profinite sets T_1, T, T_2 and maps $\alpha_i\colon T_i \to T$ we have*

$$\underline{T_1 \times_T T_2} = \underline{T_1} \times_{\underline{T}} \underline{T_2}.$$

Proof. By the basic properties of fibered products, for any profinite set S we have

$$\mathrm{Hom}(S, T_1 \times_T T_2) = \mathrm{Hom}(S, T_1) \times_{\mathrm{Hom}(S,T)} \mathrm{Hom}(S, T_2).$$

□

Technical Aside. For set theoretic reasons one needs to work in a universe where the cardinality of the profinite sets under consideration is bounded. Fix an uncountable strong limit cardinal κ (*strong limit cardinal* meaning for any cardinal $C < \kappa$ it is also true that $2^C < \kappa$). We consider the category $\mathbf{Prof}_\kappa$ of profinite sets of cardinality less than that of κ, and the resulting category of κ-condensed sets, groups, etc. Later we shall see the relationship between the various $\mathbf{Prof}_\kappa$, and eventually we will take a limit over all strong limit cardinals. Here we remark that all the results in this section hold in $\mathbf{Prof}_\kappa$ for any strong limit cardinal κ. The resulting category of sheaves on this Grothendieck site is called the category of κ-condensed sets.

2.2 Quasi-compact Condensed Sets

In this section we fix a strong limit cardinal κ. A map between κ-condensed sets $f\colon X \to Y$ is *surjective* if for evey profinite set S and an element $\alpha \in Y(S)$ there is a covering $\{\varphi_i\colon S_i \to S\}_i$ such $\varphi_i^*\alpha \in Y(S_i)$ is in the image of $f(S_i)\colon X(S_i) \to Y(S_i)$ for each i. A map $f\colon X \to Y$ of κ-condensed sets is said to be *imjective* if for every profinite set S the map $f(S)\colon X(S) \to Y(S)$ is injective.

Definition 2.4. A κ-condensed set T is *quasi-compact (qc)* if for any surjection (in the category of condensed sets) $\coprod_{i\in I} T_i \to T$ there is a finite subset $\{i_1, \ldots, i_k\} \subset I$ such lhat $\coprod_{j=1}^k T_{i_j} \to T$ is surjective.

Proposition 2.5. *A κ-condensed set T is qc if and only if there is a κ-condensed set S and a surjection $\underline{S} \to T$.*

Proof. We begin with a lemma.

Lemma 2.6. *Suppose we have a morphism of κ-condensed sets $f\colon T \to \underline{S}$. Then f is a surjection if and only if there is a covering $\{\varphi_i\colon S_i \to S\}_{i\in I}$ such that $\alpha_i \in \varphi_i^*(\mathrm{Id}_S) \in \underline{S}(S_i)$ is the image of $f(S_i)(\beta_i)$ for some $\beta_i \in T(S_i)$ for every $i \in I$*

Proof. Necessity is obvious. Let us show the condition is sufficient to establish surjectivity. Given a κ-condensed set S' and $\gamma \in \underline{S}(S')$, the element γ corresponds to a map $\gamma\colon S' \to S$ that pulls back Id_S to $\gamma \in \underline{S}(S')$. Form the diagram

$$\begin{array}{ccc} U \times_S S_i & \xrightarrow{\pi} & S_i \\ {\scriptstyle g_i}\big\downarrow & & \big\downarrow{\scriptstyle f_i} \\ S' & \xrightarrow{\gamma} & S. \end{array}$$

We see that $\pi_2^*\alpha_i = g_i^*\gamma$ and $\pi_2^*\alpha_i$ is the image of $\pi_2^*\beta_i$. This proves that f is surjective. □

Lemma 2.7. *Any representable κ-condensed $\underline{S}$ is qc.*

Proof. Let $f\colon \coprod_i T_i \to \underline{S}$ be a surjection. By surjectivity there is a covering $\varphi_j\colon S_j \to S_j$ such that setting $\alpha_j = \varphi_j^*(\mathrm{Id}_S)$, there are elements in $\beta_j \in \coprod_i T_i(S_j)$ with $f(S_j)(\beta_j) = \alpha_j$. We must show that there is a finite co-product of the T_i that contains all the elements β_j for then by the previous lemma the restriction of f to this finite co-product will also be a surjection. Of course, it suffices to prove that each β_j is contained in a finite co-product because the finite co-product of these finite co-produdts will contain all the β_j.

Consider the element $\beta_j \in \coprod_i T_i(S_j)$. By the definition of the co-product this means that there is a covering $\{f_k\colon S_k \to S_j\}_{j=1}^n$ and an element in $(\gamma_1, \ldots, \gamma_n) \in \prod_{k=1}^n \left(\coprod_i T_i(S_k)\right)$ which is in the equalizer of two pull backs to $\prod_{k,k'} \left(\coprod_i T_i(S_k \times_{S_j} S_{k'})\right)$ and which represents β_j in the co-product of such equalizers, which is the construction of the sheafification of the co-product evaluated on S_j. But each γ_k lies in $X_{i(k)}(S_k)$ for some $i(k)$. Thus, $(\gamma_1, \ldots, \gamma_k)$ represents an element in $\coprod_{i\in I} X_i(S_j)$ where $I = \{i(1), \ldots, i(n)\}$. This element maps to β_j in the full co-product so that its image in $\underline{S}(S_j)$ is α_j.

This completes the proof of the existence of a fine sub-co-product mapping surjectively to $\underline{S}$. This proves that $\underline{S}$ is qc. □

We can now establish one direction of the proposition.

Corollary 2.8. *The quotient of a representable κ-condensed set is qc.*

Proof. Let T be a κ-condensed set that is the quotient of a representable κ-condensed set $\underline{S} \to T$. Let $\coprod_i X_i \to T$ be a a surjection. Then $\underline{S} \times_T \coprod_i X_i \to \underline{S}$ is a surjection. Also,

$$\underline{S} \times_T \coprod_i X_i = \coprod_i (\underline{S} \times_T X_i).$$

By what we just proved there is a finite co-product $\coprod_{i \in I} \underline{S} \times_T X_i$ so that the restriction of the map to this sub-co-product is a surjectively map to X. It follows that the restriction of the map of $\coprod_i X_i \to T$ to the finite co-product $\coprod_{i \in I} X_i$ is a surjective map to T. □

Now we consider the converse. We suppose that T is qc.

Claim 2.9. *Each κ-condensed set T is the image of a co-product (in the category) of representable κ-condensed sets.*

Proof. For each κ-profinite set S and each $\alpha \in T(S)$ there is a morphism $\underline{S} \to T$ that sends Id_S to α. Thus, we take the set of one of κ-condensed set from each isomorphism class, $\{S_i\}$, and for each S_i the set of elements $\alpha_{i,j} \in T(S_i)$. For each $\alpha_{i,j}$ we have a morphism $a_{i,j} \colon \underline{S}_i \to T$ such that $a_{i,j}(\mathrm{Id}_{S_i}) \in \underline{S_i}(S_i) = \alpha_{i,j} \in T(S_i)$. The co-product of these maps as we range over all S_i and for each S_i range over all $\alpha \in T(S_i)$ is surjective. This proves that every κ-condensed set is the image of a co-product of representable κ-condensed sets. □

Take a surjection $\coprod_i \underline{S}_i \to T$. Since T is qc, there is a surjection from $\coprod_{j=1}^k \underline{S}_{i(j)} \to T$, meaning that there is a surjection from $\underline{S_{i(1)} \coprod \cdots \coprod S_{i(k)}} \to T$. This completes the proof of the proposition. □

2.3 Quasi separated condensed sets

Definition 2.10. A κ-condensed set T is *quasi-separated (qs)* if the diagonal map is quasi-compact; i.e., if for any pair of maps $f, g \colon \underline{S} \to T$ the fibered product $S \times_T S$ is quasi-compact.

Proposition 2.11. *T is quasi-separated if and only if for any representable κ-condensed set, $\underline{S}$, and any pair of maps $f, g \colon \underline{S} \to T$ the product $S \times_T S$ is represented by a closed subset of $S \times S$.*

Proof. The 'if' direction is clear. Consider the 'only if' direction. This follows immediately from the more general statement in the following lemma.

Lemma 2.12. *Let T be a subsheaf of $\underline{S}$ for some κ-condensed set S. Then T is qc if and only if T is represented by a closed subset of S.*

Proof. Again the 'if' direction is immediate, and we are left to suppose that $T \subset \underline{S}$ is qc. Since $T \subset \underline{S}$, for any $U \in \mathbf{Prof}_\kappa$ the elements of $T(U)$ are identified with continuous functions $U \to S$. We are assuming that T is qc, so there is a surjective map $\underline{S'} \to T$, which is a continous map $f : S' \to S$. Since $\underline{S'} \to T$ is surjective for any $U \in \mathbf{Prof}_\kappa$ there is a covering $\{U_j \to U\}$ such that for each j the map $h_j = h|_{U_j}$ lifts to a map $\widetilde{h} : U_j \to S'$. Lifting means that $f \circ \widetilde{h}_j = h_j$. In particular the image of h_j is contained in the closed subset $f(S') \subset S$. Since this is true for all h_j it is also true that the image $h(U) \subset f(S')$. Since U is an arbitrary κ-condensed set and h is an arbitrary element of $T(U)$, it follows that $T \subset \underline{f(S')}$. On the other hand, we have the composition $\underline{S'} \to T \to \underline{f(S')}$ which is $\underline{f} : \underline{S'} \to \underline{f(S')}$.

Claim 2.13. *$\underline{f} : \underline{S'} \to \underline{S}$ is surjective.*

Proof. Since $f : S' \to f(S')$ is a surjection, it is a covering in the Grothendieck topology. The pull back by this map of $\mathrm{Id}_{f(S')}$ is $f \in \underline{S}(S')$. Clearly this is the $\underline{f}(\mathrm{Id}_{S'})$. Invoking Lemma 2.6 we see that $\underline{f}$ is surjective. □

Since $\underline{f}$ is surjective and factors though the inclusion $T \to \underline{f(S')}$, this implies that $T \to \underline{f(S')}$ is also surjective and hence that $T \to \overline{\underline{f(S')}}$ is an isomoprhism of condensed sets. □

This completes the proof of the proposition. □

2.4 Weak Topologies

Given any space X we can define a second topology on X, the compactly generated topology on X. Consider the compact Hausdorff spaces K with continuous maps to X. In the compactly generated topology and subset of X is open if and only if its pre-image under each map from a compact Hausdorff space to X that is continuous in the original topology is open. It is easy to see that this forms a (possibly new) topology on X, denoted X^{cg}. Since every open set in the original topology is an open subset in the compactly generated topology, the identity map $X^{cg} \to X$ is continuous, but not necessarily a homeomorphism. Nevertheless, the induced map $\underline{X}^{cg} \to \underline{X}$ is an isomorphism of sheaves since maps from a profinite spae (which is a compact Hausdorff) into X is continuous using one of the topologies on X if and only if it is continuous using the other.

We can describe the compactly generated topology in another way. Consider all continuous maps $S \to X$ where S is a profinite space. We define the weak topology on X determined by this family of maps. It is the topology whose open subsets are exactly those whose re-image under each map from a profinite set is open in that finite set. This topology is the same as the compactly generated topology since every compact Hausdorff space is the quotient of a profinite space.

We can also put bound on the cardinality of the compact Hausdorff spaces or profinite sets used to defining the weak topology. For any cardinality κ we define the κ-compactly generated topology to the be topology defined by continuous maps (meaning continuous with respect to the given topology) from compact Hausdorff spaces of cardinality less than κ to X. This κ-compactly generated topology on X is denoted $X^{\kappa-\mathrm{cg}}$.

We can also consider the weak topology defined by all continuous maps of profinite sets of cardinality less than κ to X. If κ is a strong limit cardinal, then these two topologies agree. The reason is that an examination of the proof that every compact Hausdorff space is the quotient of an extremely disconnected set shows that if the cardinality of the compact Hausdorff space is C, then the cardinality of the extremely disconnected set mapping onto it can be taken to be 2^{2^C}. But if κ is a strong limit cardinal then $C < \kappa$ implies $2^{2^C} < \kappa$.

Proposition 2.14. *The map $X \mapsto \underline{X}$ from topological spaces to κ-condensed sets is a faithful functor. It is fully faithful on the full subcategorty of κ-compactly generated spaces. Its left adjoint is the map is given by $\mathcal{A} \mapsto \mathcal{A}(*)$. equipped with the quotient topology of $\coprod_{\underline{S}\to\mathcal{A}} S \to \mathcal{A}(*)$ where $\coprod_{\underline{S}\to\mathcal{A}}$ runs over all maps of all profinite sets of cardinality less than κ. The natural map $\underline{X}(*) \to X$ is the inclusion X with its κ-compactly generated topology into X with its original topology.*

Proof. A continuous map $f\colon X \to Y$ gives rise to $\underline{f}\colon \underline{X} \to \underline{Y}$ and the map f is recovered as the set function $\underline{f}(*)$, showing that this functor is faithful.

Fix a topological space X and a condensed set A. Then $A(*)$ is the value of A on the one-point space. We give it a weak topology as follows. For each profinite set S of cardinality less than κ recall that the set of morphisms $\underline{S} \to A$ is identified with $A(S)$ by sending a morphism ψ to $\psi(\mathrm{Id}_S) \in A(S)$. On the other hand, a morphism $\psi\colon \underline{S} \to A$ induces a set function $\psi(*)\colon S \to A(*)$. We say that a subset U of $A(*)$ is open if and only if for every profinite set S of cardinality less than κ and every element $\alpha \in A(S)$ the pre-image of U under the resulting function $\psi_\alpha(*)\colon S \to A(*)$ is an open subset of S.

This clearly defines a topology on $A(*)$, the weak topology determined by all elements of $A(S)$ for all profinite sets S of cardinality less than κ.

Let us define maps both ways between $\mathrm{Hom}_{\mathrm{top}} A(*), X)$ and $\mathrm{Morph}_{\kappa-\mathrm{cond}}(A, \underline{X})$, Fix a continuous map $\varphi\colon A(*) \to X$. For a profinite set S of cardinality less than κ we define $\varphi(S)\colon A(S) \to \underline{X}(S)$ as follows. For each $\alpha \in A(S)$ we have the associated function $\alpha(*)\colon S \to A(*)$ which is continuous by the definition of the topology on $A(*)$. We let $\varphi(S)(\alpha)$ be the composition $\varphi(S) \circ \alpha(*)$. Being a composition of continuous maps it is continuous and hence is an element of $\underline{X}(S)$. Fuctorality of φ in S is immediate, so we have defined a morphism in κ-condensed sets from A to $\underline{X}$.

In the opposite direction, given a morphism of κ-condensed sets $\psi\colon A \to \underline{X}$ we associate the function $\psi(*)\colon A(*) \to X$. To prove continuity of $\psi(*)$ is to show that $\psi(*)^{-1}(U)$ is open for every open set $U \subset X$. Fix an open subset $U \subset X$. To show $\psi(*)^{-1}(U)$ is open we need to show that for every profinite set S of cardinality less than κ and every $\alpha \in A(S)$ under the resulting map $\alpha(*)\colon S \to A(*)$ the pre-image of $\psi(*)^{-1}(U)$ is open. But since ψ is a morphism of κ-condensed sets $\psi(\alpha) \in \underline{X}(S)$. The map $S \to X$ that results from $\psi(\alpha)$ is $\psi(*) \circ \alpha(*)$. Since $\psi(\alpha)$ is an element of $\underline{X}(S)$, it follows that $\psi(*)\circ\alpha(*)$ is continuous and hence the pre-image of U under this map is an open subset of S. It follows immediately the the pre-image under $\alpha(*)$ of $\psi(*)^{-1}(U)$ is open in S. Since this is true for all profinite sets S of cardinality less than κ and all $\alpha \in A(S)$, it follows from the definition of the topology on $A(*)$ that $\psi(*)^{-1}(U)$ is open, proving that $\psi(*)$ is continuous.

It is direct to see that the functions are inverses of each other.

Now if X has a κ-compactly generated topology, then, since every κ-compact Hausdorff space is the quotient of a profinite space of cardinality less than κ, it also has the weak topology generated by all continuous maps $S \to X$ for S ranging over profinite spaces of cardinality less than κ. In this case the topology of X and $X(*)$ agree and we see that $X \to \underline{X}$ is fully faithful on the subcategory of κ-compactly generated spaces. $\square$