INTRODUCTION TO ANALYTIC RINGS

One of the major breakthroughs of Clausen and Scholze's condensed mathematics is the introduction of analytic rings. These objects generalise essentially any kind of algebraic and analytic geometry that appears in practice, such as classical and formal schemes, topological and differential manifolds, or rigid and complex analytic spaces. In this lecture we motivate the definition of analytic rings, and provide some current applications in geometry. We do not pretend to give a full historical justification of the definition of an analytic ring, the heuristics down below come from the experience of working with those objects, any error or misconception in the actual philosophical meaning of an analytic ring is due completely to the author. This talk is mostly expository and essentially no proof will be given.

1. Why analytic rings?

One of the dreams of algebraic geometers is to have a framework where *all* kind of different geometries appearing in mathematics live together. Since *all* geometries in mathematics is a vague object, let us pretend the geometries we care are given by differential or topological manifolds, algebraic varieties and schemes, archimedean and non-archimedean analytic geometries such as complex and rigid analytic geometry, generalizations of these last two such as Berkovich or adic spaces. A question arises, what does it mean to be given with one of the objects on such categories? What is the important data that determines those different geometries? In this section we will see how the concept of analytic ring captures the correct information. First, let us make a brief summary of the notions we want to generalize.

(1) A point in common that all geometries share is a well defined sheaf of functions. Let X be a nice topological space, say a CW complex with finitely many cells of given dimension. There are different ways to see X as a geometric object. First, for any topological field K (eg. $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) one can consider the sheaf $\mathscr{O}_{X,K}$ of continuous functions $X \to K$, it gives a an incarnation as a "topological K-manifold". We can also consider the most basic sheaf $\mathscr{O}_{X,\mathbb{Z}}$ consisting on $\mathbb{Z}$-valued locally constant functions on X, this is the initial sheaf on rings over the topological space X, so there is no harm saying that it is the *defines the topological space X as a geometric object*. In other words, any other possible possible geometry we put in X has a natural map towards the geometry $(X, \mathscr{O}_{X,\mathbb{Z}})$.

(2) In differential geometry, a smooth manifold is defined as a topological space X together with smooth charts into open subsets of $\mathbb{R}^n$. An equivalent way to provide this data is by endowing X with the sheaf of smooth functions $\mathscr{O}_X$. Indeed, given $U \subset X$ an open subspace diffeomorphic to an open ball in $\mathbb{R}^n$, we can recover the topological space U by looking at the maximal ideals of $\mathscr{O}_X(U)$, and by looking at the topology generated by inequalities of the form $|f| \leq 1$ or $1 \leq |f|$ for $f \in \mathscr{O}_X(U)$.

(3) In algebraic geometry the relation between points and rings is more notorious. By definition, an affine scheme is a commutative ring seen in its opposite category. From a commutative ring R we can construct its Zariski spectrum $\operatorname{Spec} R$ consisting on prime ideals, and topology given by subsets of the form $f \neq 0$. Then, one constructs a geometric object $(\operatorname{Spec} R, \mathscr{O})$ which we usually refer as the spectrum of R. General schemes are constructed by glueing together affine schemes using the Zariski topology. Note that the ring R already carries all the relevant information; the construction of $\operatorname{Spec} R$ and the Zariski topology are arbitrary choices made in order to realize R as a geometric object.

(4) In complex and rigid analytic geometry the treatment is similar as for schemes. There is a class of good rings R (eg. holomorphic functions on open balls of $\mathbb{C}^n$ or the so called Tate algebras). From the rings R one can construct their spectrum given by suitable points that only depend on the ring R (for complex spaces these are algebra homomorphisms $R \to \mathbb{C}$, for rigid spaces one can take either maximal ideals or continuous valuations depending on the taste of the formalism). Then, one defines analytic topologies by taking subspaces attached to inequalities $|f| \leq 1$ and $1 \leq |f|$, and gluing along these analytic topologies defines complex or rigid analytic varieties.

We can conclude from the previous examples that, in order to do geometry, we only need to know the rings of functions of the spaces, and remember how they glue forming an actual geometric space. Following the example of algebraic geometry, the first part of the data is provided by the category of commutative rings Ring, and its opposite category Aff of affine schemes. Then, the most general way we can remember how rings glue together and form geometric spaces is by looking their functors of points, namely, looking at the category of presheaves $\mathrm{PSh}(\mathrm{Aff}, \mathrm{Sets})$. In fact, the category $\mathrm{PSh}(\mathrm{Aff}, \mathrm{Sets})$ is obtained from Aff by "freely" adding colimits, i.e. it is the initial object in the category of morphisms $\mathrm{Aff} \to \mathscr{C}$ such that $\mathscr{C}$ has all colimits. Thus, given a presheaf T written as a colimit $T = \varinjlim_I \operatorname{Spec} R_i$, the *shape of the diagram* $I \to \mathrm{Aff}$ tells us how to glue the affine schemes to the object T. Finally, depending on an arbitrary choice that can be more convenient in some situations, we can fix a Grothendieck topology $\mathcal{T}$ on Aff (eg. the Zariski/étale/fppf topologies), and identify different presheaves by sheafifying, or equivalently, by looking at the category of $\mathcal{T}$-sheaves on Aff, namely, $\mathrm{Sh}_{\mathcal{T}}(\mathrm{Aff}, \mathrm{Sets})$. By restricting to a certain category of sheaves we allow more gluings of affine schemes in less objects (many presheaves will identify to the same sheaf after sheafification), which in practice translates in capturing locally the relevant information we want to study.

Generalizing in other direction, cohomological invariants in algebraic geometry have a derived flavour; just the notion of cohomology involves working in the context of derived categories. In order to have a better behaviour of the theory (and also inspired from homotopy theoretic methods in algebraic topology), one must work with higher category theory, and replace the category of commutative rings by its left derived category (in the sense of Quillen) which nowadays we call the category of *animated rings* AniRing, see §2. Similarly, we must replace the category of sets by its animation, the category of *anima* (eq. spaces or ∞-groupoids), and the category of sheaves on sets by the category of sheaves on anima.

So far we have just justified the introduction of (derived) algebraic geometry. But, how does analytic geometry appear? Well, strictly speaking classical algebraic geometry does not cover the situations (1), (2) and (4) previously mentioned: there is not a way to capture analytic topologies just by doing classical algebra since we need to deal with actual topological rings, and subspaces defined by inequalities instead of non-vanishing functions. Thanks to condensed mathematics we can solve the first issue by replacing (animated) rings by the category of condensed (animated) rings. However, how do we capture different localizations of a condensed ring arising from subsets of the underlying geometric space? We can solve this problem by making the following observation: when doing geometry we do not just focus on the rings themselves, we mostly consider different sheaves of modules that we like to call "quasi-coherent sheaves", they are one of the most important algebraic invariants we study in any kind of geometry as they encode most of the cohomological information of our spaces. In algebraic geometry the quasi-coherent sheaves satisfy the following pleasant properties for doing homological algebra:

- The category of (derived) quasi-coherent sheaves on a scheme is a stable ∞-category admitting all small limits and colimits.
- The category of (derived) quasi-coherent sheaves has a closed symmetric monoidal structure, i.e. a tensor product $- \otimes^L -$ and an internal $R\underline{\mathrm{Hom}}$.

Moreover, the structural sheaf $\mathscr{O}_X$ of a scheme is determined by the category of quasi-coherent modules $\mathscr{D}(X)$ as the unit of the symmetric monoidal structure. This suggests that, in order to define the correct notion of localization for condensed rings that will allow to construct several instances of analytic spaces, we can try to localize the symmetric monoidal category of condensed modules.

Summarizing the previous (long) discussion, the notion of "analytic ring" we want to define should be given by the data of a condensed (animated) ring $\underline{\mathcal{A}}$ which plays the role of the topological ring of functions of our space, together with a suitable stable closed symmetric monoidal category $\mathscr{D}(\mathcal{A})$ which ought to be a localization of the category of condensed $\underline{\mathcal{A}}$-modules $\mathscr{D}(\underline{\mathcal{A}})$. The beauty of the definition of analytic rings is that, even though we arrived to the previous first approximation by trying to solve the desired properties that an analytic geometry theory should satisfy (i.e. spaces determined by "topological rings" admitting general localizations for defining "analytic subspaces"), the actual definition is not far from the previous and still has an interpretation in terms of measures and complete modules. In the next section we give the formal definition of analytic rings and mention (without many details) some applications of the theory in actual geometry, in particular, we will mention the most important analytic rings (and essentially the only ones we know so far), namely, the solid and liquid rings.

2. Animation and condensification

We follow [CS20, Lecture XI]. The process of animation of a category dates back to Quillen with his non-abelian derived categories, and as reference one can take [Lur09, §5.5.8]; the name of animation was suggested by Clausen, and used first in [CS23].

Definition 2.0.1. Let $\mathscr{C}$ be a (1-)category that admits small colimits.

(1) An object $X \in \mathscr{C}$ is called compact if the functor $\mathrm{Hom}(X, -)$ commutes with filtered colimits.
(2) An object $X \in \mathscr{C}$ is called projective if $\mathrm{Hom}(X, -)$ commutes with reflexive coequalizers, i.e. colimits of $\Delta^{op}_{\leq 1}$ diagrams (coequalizers of two arrows $Y \rightrightarrows Z$ with a common section $Z \to Y$).

We let $\mathscr{C}^c$, $\mathscr{C}^p$ and $\mathscr{C}^{cp}$ denote the full subcategories of $\mathscr{C}$ consisting on compact, projective, and compact projective objects.

There is a fully faithful embedding of $1{-}\mathrm{sInd}(\mathscr{C}^{cp}) \to \mathscr{C}$ from the 1-sifted Ind category of $\mathscr{C}^{cp}$ (i.e. the category freely generated under 1-sifted colimits by the Yoneda embedding, equivalently, the full subcategory of $\mathrm{PSh}(\mathscr{C}^{cp}, \mathrm{Sets})$ that commutes with finite products). If $\mathscr{C}$ is generated under colimits by $\mathscr{C}^{cp}$, we have an equivalence

$$1{-}\mathrm{sInd}(\mathscr{C}^{cp}) \cong \mathscr{C}.$$

Example 2.0.2. The objects in $1{-}\mathrm{sInd}(\mathscr{C}^{cp})$ have a "concrete description" thanks to (the 1-categorical version of) [Lur09, Lemma 5.5.8.14]. By definition, $1{-}\mathrm{sInd}(\mathscr{C}^{cp})$ is the full subcategory of $\mathrm{PSh}(\mathscr{C}^{cp}, \mathrm{Sets})$ that commute with finite coproducts. A presheaf $F \in \mathrm{PSh}(\mathscr{C}^{cp}, \mathrm{Sets})$ is in $1\,\mathrm{sInd}(-\mathscr{C}^{cp})$ if and only if there is a (≤ 1)-simplicial object $U_1 \rightrightarrows U_0 \to U_1$ in $\mathrm{Ind}(\mathscr{C}^{cp})$ whose coequalizer is F. Conversely, the category $\mathrm{sInd}(\mathscr{C}^{cp})$ is stable under reflexive coequalizers and filtered colimits.

Remark 2.0.3. For an ∞-category $\mathscr{C}$ with small colimits we can also define compact and projective object; compact objects are those X such that $\mathrm{Hom}(X, -)$ commutes with (homotopy) filtered colimits, and projective objects are those X for which $\mathrm{Hom}(X, -)$ commutes with geometric realizations, i.e. Δ^{op}-diagrams. We can still form the objects $\mathscr{C}^c$, $\mathscr{C}^p$ and $\mathscr{C}^{cp}$. In this situation, we define $\mathrm{sInd}(\mathscr{C}^{cp})$ to be the sifted Ind category of $\mathscr{C}^{cp}$ obtained by freely adjoining sifted colimits of objects of $\mathscr{C}^{cp}$, or equivalently, as the full subcategory of $\mathrm{PSh}(\mathscr{C}^{cp}, \mathrm{Ani})$ which preserves finite products. By [Lur09, Lemma 5.5.8.14], a presheaf $F \in \mathrm{PSh}(\mathscr{C}^{op}, \mathrm{Ani})$ is in $\mathrm{sInd}(\mathscr{C}^{cp})$ if and only if there is a simplicial diagram $U_\bullet$ in $\mathrm{Ind}(\mathscr{C}^{op})$ whose geometric realization is F. Conversely, $\mathrm{sInd}(\mathscr{C}^{cp})$ is stable under filtered colimits and geometric realizations.

Example 2.0.4 ([CS20, Example 11.3])**.** The following categories are compactly generated and we make explicit their compact projective generators.

(1) For $\mathscr{C} = \mathrm{Sets}$ the category of sets, $\mathscr{C}^{cp}$ is the category of finite sets.
(2) For $\mathscr{C} = \mathrm{Ab}$ the category of abelian groups, $\mathscr{C}^{cp}$ is the category of finite free abelian groups.
(3) For $\mathscr{C} = \mathrm{Ring}$ the category of commutative rings, $\mathscr{C}^{cp}$ is the category of retracts of polynomial algebras in finitely many variables.
(4) For $\mathscr{C} = \mathrm{Cond}$ the category of condensed sets, $\mathscr{C}^{cp}$ is the category of extremally disconnected sets.
(5) For $\mathscr{C} = \mathrm{Cond}(\mathrm{Ab})$ the category of condensed abelian groups, $\mathscr{C}^{cp}$ is the category of retracts of the free condensed abelian groups $\mathbb{Z}[S]$ with S extremally disconnected.
(6) For $\mathscr{C} = \mathrm{Cond}(\mathrm{Ring})$ the category of condensed rings, $\mathscr{C}^{cp}$ is the category of retracts of the condensed commutative rings generated by extremally disconnected sets, i.e., retracts of rings of the form $\mathbb{Z}[\mathbb{N}[S]]$ with S extremally disconnected.

The process of animation (or taking left derived categories) consists on adding arbitrary sifted colimits to the full subcategory of compact projective objects of a 1-category.

Definition 2.0.5. Let $\mathscr{C}$ be a 1-category with small colimits, its animation is the category $\mathrm{Ani}(\mathscr{C}) := \mathrm{sInd}(\mathscr{C}^{cp})$.

Example 2.0.6. (1) $\mathrm{Ani}(\mathrm{Sets})$ is the ∞-category of anima which we simply write by Ani.

(2) $\mathrm{Ani}(\mathrm{Ab})$ is the ∞-category of animated abelian groups, by the Dold-Kan correspondence it is equivalence to the connective ∞-derived category of abelian groups.
(3) $\mathrm{Ani}(\mathrm{Ring})$ is the ∞-derived category of animated rings (or simplicial rings in more classical literature). If we work over $\mathbb{Q}$, $\mathrm{Ani}(\mathrm{Ring}_{\mathbb{Q}})$ is equivalent, via the Dold-Kan correspondence, to the ∞-category of connective differential graded algebras over $\mathbb{Q}$.

In contrast to animation, we can also define condensed objects in a category $\mathscr{C}$.

Definition 2.0.7 ([CS20, Definition 11.7], [Man22b, Definition 2.1.1])**.** Let $\mathscr{C}$ be an ∞-category with small colimits and finite limits. Let κ be an uncountable strong limit cardinal. The κ-small condensed objects of $\mathscr{C}$ are defined as the ∞-category $\mathrm{Cond}_\kappa(\mathscr{C})$ of functors $F : \mathrm{Extdis}_\kappa^{op} \to \mathscr{C}$ mapping co-products to products. We let $\mathrm{Cond}(\mathscr{C})$ be the colimit

$$\mathrm{Cond}(\mathscr{C}) = \varinjlim_{\kappa} \mathrm{Cond}_\kappa(\mathscr{C})$$

along the fully faithful left adjoints of the restriction functors.

By [CS20, Lemma 11.8], if $\mathscr{C}$ is a compactly generated 1-category with small colimits, then there is a natural equivalence

$$\mathrm{Cond}(\mathrm{Ani}(\mathscr{C})) \cong \mathrm{Ani}(\mathrm{Cond}(\mathscr{C})).$$

In particular, we can let CondAni denote either Ani(Cond) or Cond(Ani) and call it the ∞-category of condensed anima.

3. Analytic rings

After the reminders in left derived categories or animation, we are in good shape to define analytic rings. Recall that after the discussion in the introduction an analytic ring should be given by an animated condensed ring $\underline{\mathcal{A}}$ together with a suitable localization $\mathscr{D}(\mathcal{A})$ of the derived category $\mathscr{D}(\underline{\mathcal{A}})$ of condensed $\mathcal{A}$-modules. By Lectures II-III, the category of condensed abelian groups has compact projective generators given by $\mathbb{Z}[S]$ with $S \in \mathrm{Extdis}$. This implies that the category of connective $\underline{\mathcal{A}}$-modules $\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ is also generated by the compact projective objects $\underline{\mathcal{A}}[S]$ under sifted colimits. So, in order to keep the animated nature of the category of modules of $\underline{\mathcal{A}}$, we could ask for localizations of the symmetric monoidal category $\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$; by stabilization we would obtain localizations of $\mathscr{D}(\underline{\mathcal{A}})$ itself. For such a localization $\mathscr{D}_{\geq 0}(\mathcal{A})$, we should have a projection functor

$$\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} - : \mathscr{D}_{\geq 0}(\underline{\mathcal{A}}) \to \mathscr{D}_{\geq 0}(\mathcal{A})$$

with fully faithful right adjoint

$$F : \mathscr{D}_{\geq 0}(\mathcal{A}) \to \mathscr{D}_{\geq 0}(\underline{\mathcal{A}}).$$

We can the ask for the category $\mathscr{D}_{\geq 0}(\mathcal{A})$ to be compactly generated and for the base change functor $\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} -$ to preserve compact projective generators. In other words, we want $\mathscr{D}_{\geq 0}(\mathcal{A})$ to be the generated under sifted colimits by the objects $\mathcal{A}[S] := \mathcal{A} \otimes^L_{\underline{\mathcal{A}}} \underline{\mathcal{A}}[S]$. Thus, F needs to commute with colimits. On the other hand, we want $\mathscr{D}_{\geq 0}(\mathcal{A})$ to be enriched in condensed anima in order to define an internal Hom, and that the functor F is a fully faithful immersion of categories enriched in condensed anima. This implies in particular that, if $M \in \mathscr{D}_{\geq 0}(\mathcal{A})$ and $S \in \mathrm{Extdis}$, then $\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[S], M)$ should also be in $\mathscr{D}_{\geq 0}(\mathcal{A})$. This leads to the following definition.

Definition 3.0.1 ([CS20, Proposition 12.20])**.** An associative analytic ring $\mathcal{A}$ is a pair consisting on a animated condensed associative ring $\underline{\mathcal{A}}$ and a full subcategory $\mathscr{D}_{\geq 0}(\mathcal{A}) \subset \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ satisfying the following properties:

1. $\mathscr{D}_{\geq 0}(\mathcal{A})$ is stable under all limits and colimits.
2. $\mathscr{D}_{\geq 0}(\mathcal{A})$ is stable under $\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\mathbb{Z})}(\mathbb{Z}[S], -)$ for $S \in \mathrm{Extdis}$.
3. The inclusion $\mathscr{D}_{\geq 0}(\mathcal{A}) \subset \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ has a left adjoint (which we denote by $\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} -$).

We call $\mathscr{D}_{\geq 0}(\mathcal{A})$ the ∞-category of animated $\mathcal{A}$-complete modules.

Remark 3.0.2. In Definition 3.0.1, the adjoint functor theorem guarantees the existence of the left adjoint $\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} -$ by (1) modulo set theory (we need the inclusion to the an accessible functor).

Remark 3.0.3. In Definition 3.0.1 we could replace animated condensed rings by $\mathbb{E}_1$-condensed rings and the category of animated condensed modules by the full stable category of condensed modules $\mathscr{D}$. Then properties (1)-(3) still make sense, and we can create an even larger class of *spectral associative analytic rings*.

Of course, to give the data of a full subcategory of animated condensed $\underline{\mathcal{A}}$-modules is not easy. A better way to encode this category is via its compact projective generators, this leads to the original definition of an analytic ring

Definition 3.0.4 ([CS20, Definition 12.1])**.** An associative analytic ring $\mathcal{A}$ is the data of an associative animated condensed ring $\underline{\mathcal{A}}$, together with a functor $\mathcal{A}[-] : \mathrm{Extdis} \to \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ and a natural transformation $S \to \mathcal{A}[S]$, satisfying the following conditions:

(1) $\mathcal{A}[-]$ sends finite coproducts to direct sums.
(2) For any $C \in \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ which is a sifted colimit of objects of the form $\mathcal{A}[S]$ for $S \in \mathrm{Extdis}$, the natural map

$$\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathcal{A}[S], C) \to \underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathbb{Z}})}(\mathbb{Z}[S], C)$$

is an equivalence.

We say that an analytic ring $\mathcal{A}$ is *complete* if the natural map $\underline{\mathcal{A}} \to \mathcal{A}[*]$ is an equivalence.

We let $\mathscr{D}_{\geq 0}(\mathcal{A})$ be the full subcategory of $\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ spanned by those objects C such that the natural map

$$\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathcal{A}[S], C) \to \underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathbb{Z}})}(\mathbb{Z}[S], C)$$

is an equivalence for all $S \in \mathrm{Extdis}$.

The main properties of the categories of analytic rings are summarized in the following lemma.

Proposition 3.0.5 ([CS20, Proposition 12.4], [Man22b, Proposition 2.3.2])**.** *Let $\mathcal{A}$ be an associative analytic ring. The ∞-category $\mathscr{D}_{\geq 0}(\mathcal{A})$ is generated under sifted colimits by the objects $\mathcal{A}[S]$ for $S \in \mathrm{Extdis}$, which are compact projective generators of $\mathscr{D}_{\geq 0}(\mathcal{A})$. The full subcategory*

$$\mathscr{D}_{\geq 0}(\mathcal{A}) \subset \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$$

is stable under all small limits and colimits and admits a left adjoint

$$\mathcal{A} \otimes^{L}_{\underline{\mathcal{A}}} - : \mathscr{D}_{\geq 0}(\underline{\mathcal{A}}) \to \mathscr{D}_{\geq 0}(\mathcal{A})$$

sending $\underline{\mathcal{A}}[S]$ to $\mathcal{A}[S]$.

The ∞-subcategory $\mathscr{D}_{\geq 0}(\mathcal{A})$ is prestable. Its heart is the abelian category $\mathscr{D}^{\heartsuit}(\mathcal{A})$ that is the full subcategory of condensed $\pi_0(\mathcal{A})$-modules generated under colimits by $\pi_0(\mathcal{A}[S])$ for $S \in \mathrm{Extdis}$. An object $C \in \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ is $\mathcal{A}$-complete if and only if all $\pi_i(C)$ lie in $\mathscr{D}^{\heartsuit}(\mathcal{A})$.

If $\underline{\mathcal{A}}$ has the structure of a condensed animated commutative ring, then there is a unique symmetric monoidal structure on $\mathscr{D}_{\geq 0}(\mathcal{A})$ such that $\mathcal{A} \otimes_{\underline{\mathcal{A}}} -$ is symmetric monoidal.

Similar statements hold true for the stabilization $\mathscr{D}(\mathcal{A})$.

Proof. By definition of analytic ring, $\mathscr{D}_{\geq 0}(\mathcal{A})$ is stable under sifted colimits of objects of the form $\mathcal{A}[S]$ for S varying. Let $\mathscr{C}^0$ be the full subcategory of $\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ consisting on objects of the form $\mathcal{A}[S]$ for varying S, and let $\mathscr{C} = \mathrm{sInd}(\mathscr{C}^0)$. Again, by definition of analytic ring, the map $\mathscr{C} \to \mathscr{D}_{\geq 0}(\mathcal{A})$ is fully faithful. Thanks to Definition 3.0.4 (2), the natural functor $\mathscr{D}_{\geq 0}(\mathcal{A}) \to \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ has a left adjoint F which uniquely extends $F(\underline{\mathcal{A}}[S]) \mapsto \mathcal{A}[S]$. The essential image of F is the category $\mathscr{C}$, which necessarily implies that $\mathscr{C} = \mathscr{D}_{\geq 0}(\mathcal{A})$, namely, we have a natural transformation $\mathrm{id} \to F$, so for $M, N \in \mathscr{D}_{\geq 0}(\mathcal{A})$ we got that

$$\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\mathcal{A})}(N, M) = \underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\mathcal{A})}(F(N), M),$$

and by Yoneda's lemma we got that $N \cong F(N)$. This shows in particular that $\mathscr{D}_{\geq 0}(\mathcal{A})$ is stable under colimits, it is also clearly stable under limits by definition.

It is formal that $\mathscr{D}_{\geq 0}(\mathcal{A})$ is prestable being closed under finite colimits and extensions in $\mathscr{D}(\underline{\mathcal{A}})$. If $C \in \mathscr{D}_{\geq 0}(\mathcal{A})$ then so is $\tau_{\geq 1} C = \Sigma \Omega C$ being the suspension of the loops of C in $\mathscr{D}_{\geq 0}(\mathcal{A})$, as it is stable under all small limits and colimits. This shows that $\pi_0(C)[0] \in \mathscr{D}_{\geq 0}(\mathcal{A})$, and one easily deduces the statement about the heart (eg. by using the Postnikov tower $C = \varprojlim_n \tau_{\leq n} C$).

Finally, if $\mathcal{A}$ is a condensed animated commutative ring, the category $\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ is symmetric monoidal, and to endow $\mathscr{D}_{\geq 0}(\mathcal{A})$ with a symmetric monoidal structure it suffices to see that the kernel of $\mathcal{A} \otimes^{L}_{\underline{\mathcal{A}}} -$ is a tensor ideal. But this kernel is generated under colimits by the cones of $\underline{\mathcal{A}}[S] \to \mathcal{A}[S]$, and for any extremally disconnected set T the tensor by $\mathbb{Z}[T]$ is still in the kernel, namely, for all $N \in \mathscr{D}_{\geq 0}(\mathcal{A})$ we have that

$$\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathrm{cone}[\underline{\mathcal{A}}[S] \to \mathcal{A}[S]], N) = 0,$$

so for any condensed set X we have

$$\mathrm{Hom}_{\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathrm{cone}[\underline{\mathcal{A}}[S] \otimes \mathbb{Z}[X] \to \mathcal{A}[S] \otimes \mathbb{Z}[X]], N) = 0.$$

Taking $X = T \times S'$ with T and S' extremally disconnected sets one deduces that

$$\underline{\mathrm{Hom}}_{\mathscr{D}_{\geq 0}(\underline{\mathcal{A}})}(\mathrm{cone}[\underline{\mathcal{A}}[S] \otimes \mathbb{Z}[T] \to \mathcal{A}[S] \otimes \mathbb{Z}[T]], N) = 0,$$

This finishes the proof. $\square$

Remark 3.0.6. Heuristically, the functor $S \mapsto \mathcal{A}[S]$ can be though as a functor of $\mathcal{A}$-valued measures on extremally disconnected sets, and the natural transformation $S \to \mathcal{A}[S]$ as the functor of Dirac measures. Thus, an $\mathcal{A}$-complete module M satisfies that any "continuous map" $f : S \to M$ extends or "integrates" uniquely to a map $\mathcal{A}[S] \to M$, meaning that for any measure $\mu \in \mathcal{A}[S]$, we can integrate $\int_S f d\mu \in M$ by taking the composite $* \xrightarrow{\mu} \mathcal{A}[S] \to M$.

Definition 3.0.7 ([Man22b, Definition 2.3.1])**.** A morphism of analytic rings $\mathcal{A} \to \mathcal{B}$ is a morphism of underlying animated condensed rings such that the forgetful functor $\mathscr{D}_{\geq 0}(\underline{\mathcal{B}}) \to \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})$ sends $\mathscr{D}_{\geq 0}(\mathcal{B})$ to $\mathscr{D}_{\geq 0}(\mathcal{A})$, i.e. such that the forgetful functor sends animated $\mathcal{B}$-complete modules to animated $\mathcal{A}$-complete modules.

Proposition 3.0.8 ([CS20, Proposition 12.6])**.** *Let $\mathcal{A} \to \mathcal{B}$ be a morphism of analytic rings. Then the forgetful functor $\mathscr{D}_{\geq 0}(\mathcal{B}) \to \mathscr{D}_{\geq 0}(\mathcal{A})$ has a left adjoint $\mathcal{B} \otimes^L_{\mathcal{A}} -$, called the analytic base change, which is the unique colimit preserving functor sending $\mathcal{A}[S]$ to $\mathcal{B}[S]$. If $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ is a morphism of condensed animated commutative rings, then $\mathcal{B} \otimes^L_{\mathcal{A}} -$ is symmetric monoidal.*

Proof. We have a commutative square of forgetful functors

$$\begin{tikzcd}
\mathscr{D}_{\geq 0}(\mathcal{B}) \arrow[r] \arrow[d] & \mathscr{D}_{\geq 0}(\mathcal{A}) \arrow[d] \\
\mathscr{D}_{\geq 0}(\underline{\mathcal{B}}) \arrow[r] & \mathscr{D}_{\geq 0}(\underline{\mathcal{A}})
\end{tikzcd}$$

whose left adjoints give rise a commutative square

$$\begin{tikzcd}
\mathscr{D}_{\geq 0}(\mathcal{B}) & \mathscr{D}_{\geq 0}(\mathcal{A}) \arrow[l, "\mathcal{B}\otimes_{\mathcal{A}}-"'] \\
\mathscr{D}_{\geq 0}(\underline{\mathcal{B}}) \arrow[u, "\mathcal{B}\otimes_{\underline{\mathcal{B}}}-"] & \mathscr{D}_{\geq 0}(\underline{\mathcal{A}}). \arrow[l, "\mathcal{B}\otimes_{\underline{\mathcal{A}}}-"'] \arrow[u, "\mathcal{A}\otimes_{\underline{\mathcal{A}}}-"']
\end{tikzcd}$$

Then the composite map $\mathcal{B} \otimes_{\underline{\mathcal{A}}} - : \mathscr{D}_{\geq 0}(\underline{\mathcal{A}}) \to \mathscr{D}_{\geq 0}(\mathcal{B})$ is the unique colimit preserving functor mapping $\underline{\mathcal{A}}[S]$ to $\mathcal{B}[S]$, the first claim follows. For the second claim, we note that the functor $\mathcal{B} \otimes_{\underline{\mathcal{A}}} -$ is symmetric monoidal by Proposition 3.0.5. It also kills the kernel of $\mathcal{A} \otimes_{\underline{\mathcal{A}}} -$ which is a tensor ideal, this implies that $\mathcal{B} \otimes_{\underline{\mathcal{A}}} -$ actually factors through a symmetric monoidal functor $\mathcal{B} \otimes_{\mathcal{A}} - : \mathscr{D}_{\geq 0}(\mathcal{A}) \to \mathscr{D}_{\geq 0}(\mathcal{B})$ as wanted. $\square$

For technical reasons, commutative analytic rings are not just given by associative analytic rings such that $\underline{\mathcal{A}}$ is endowed with a structure of commutative animated algebra. The reason is that for such a ring $\mathcal{A}$, the object $\mathcal{A}[*]$ only has the structure of a connective condensed $\mathbb{E}_\infty$-ring, and to keep working in the category of condensed animated rings we would like to $\mathcal{A}[*]$ to be endowed with a structure of condensed animated ring. An equivalent reason is that, as in derived algebraic geometry, we would like to define analytic symmetric powers functors $\mathrm{Sym}^n_{\mathcal{A}}$ just by taking base the change $\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} \mathrm{Sym}^n_{\underline{\mathcal{A}}}$. For these functors to behave in the correct way need the following definition.

Definition 3.0.9 ([CS20, Definition 12.10])**.** A commutative analytic ring $\mathcal{A}$ is an associative analytic ring $\mathcal{A}$ such that $\underline{\mathcal{A}}$ has the structure of a condensed animated commutative ring, and such that for all prime p the Frobenius map $\phi : \underline{\mathcal{A}} \to \underline{\mathcal{A}}/p$ defines a map of analytic rings.

Proposition 3.0.10 ([CS20, Lemma 12.27])**.** *Let $\mathcal{A}$ be a commutative analytic ring, then $\mathscr{D}_{\geq 0}(\mathcal{A})$ has a good theory of symmetric powers $\mathrm{Sym}^n_{\mathcal{A}} = \mathcal{A} \otimes^L_{\underline{\mathcal{A}}} \mathrm{Sym}^n_{\underline{\mathcal{A}}}$ in the sense that the analytification functor $\mathcal{A} \otimes^L_{\underline{\mathcal{A}}} -$ naturally extends to an analytification functor from animated commutative $\underline{\mathcal{A}}$-algebras to animated commutative $\underline{\mathcal{A}}$-algebras whose underlying module is $\mathcal{A}$-complete. In particular, $\mathcal{A}[*]$ has a natural structure of an animated condensed ring.*

Proposition 3.0.11 ([CS20, Proposition 12.26], [Man22b, Proposition 2.3.12])**.** *Let $\mathcal{A}$ be a commutative analytic ring, then there is an initial complete analytic ring $\mathcal{A} \to \widetilde{\mathcal{A}}$. Furthermore, it satisfies the following property:*

- *We have an equivalence of ∞-categories $\mathscr{D}_{\geq 0}(\mathcal{A}) \cong \mathscr{D}_{\geq 0}(\widetilde{\mathcal{A}})$. In particular, $\underline{\widetilde{\mathcal{A}}} = \mathcal{A}[*]$.*

4. Applications in geometry

In this last section we briefly define the two main examples of analytic rings, namely solid and liquid rings, and mention some actual applications in geometry.

Definition 4.0.1 ([Sch19, Definition 5.1])**.** We define the analytic ring of solid integers, denoted by $\mathbb{Z}_\blacksquare$, to be the analytic ring with underlying condensed ring $\mathbb{Z}$ and functor of measures given by

$$\mathbb{Z}_\blacksquare[S] = \varprojlim_i \mathbb{Z}[S_i],$$

where $S = \varprojlim_i S_i$ is written as a limit of finite sets. More generally, for a finitely generated $\mathbb{Z}$-algebra we define the analytic ring $A_\blacksquare$ by

$$A_\blacksquare[S] = \varprojlim_i A[S_i].$$

Remark 4.0.2. We will prove that $\mathbb{Z}_\blacksquare[S] = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$ is the space of measures of the locally constant functions $C(S,\mathbb{Z})$, this agrees with the idea that an analytic ring is also given by a condensed ring plus a functor of measures.

The theory of solid analytic rings (analytic rings constructed from condensed animated complete $A_\blacksquare$-algebras) is the core of nonarchimedean geometries, going from classical schemes to rigid and adic spaces. In general, this theory is used to define quasi-coherent sheaves on nonarchimedean geometries as well as theories of six functor formalisms. It has applications in p-adic Hodge theory and the local Langlands program, for example, the geometrization of the Langlands program [FS21], the construction of six functor formalisms for $\mathbb{Z}_p$ and $\mathbb{Z}_\ell$ coefficients on rigid spaces [Man22b, Man22a], the construction of analytic de Rham stacks, a rigid version of prismatic cohomology, the study of period sheaves on the v-site [Bos21], the theory of locally analytic representation of p-adic Lie groups [RJRC23], etc.

The second kind of analytic rings (and possibly the most importants in the long run), constructed in [CS20], are the liquid analytic rings $\mathbb{R}_{<p}$ (or $\mathbb{Z}((T))_{>r}$ in more generality), with $0 < p \leq 1$ a fixed real number. We leave the technical definition for a future talk. Let us just mention that these rings capture ($< p$)-locally convex analysis (in contrast with the solid analytic rings which capture (∞)-locally convex analysis, i.e. the ultrametric inequality), and they have been used to extend the theory of complex (or real) analytic varieties, to endow them with categories of liquid quasi-coherent sheaves, and to obtain a very general six functor formalism for quasi-coherent sheaves. Some classical theorems such as GAGA, Serre duality and Grothendieck-Riemann-Roch for complex varieties have been reproved in this setting. An analytic de Rham stack can be define for real analytic varieties, making the Riemann-Hilbert correspondence a tautology obtained by an equivalence $X_{dR} \cong |X|$ between the analytic de Rham stack and the underlying condensed set. The most interesting feature of the liquid theory is that it provides a clean variation of different analytic geometries interpolating both non-archimedean and archimedean geometries. We expect these rings to play a key role in the global Langlands correspondence and a global version of Hodge theory.

References

[Bos21] Guido Bosco. On the p-adic pro-étale cohomology of Drinfeld symmetric spaces, 2021.

[CS20] Dustin Clausen and Peter Scholze. Lectures on Analytic Geometry. `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`, 2020.

[CS23] Kestutis Cesnavicius and Peter Scholze. Purity for flat cohomology. To appear in Annals of Mathematics, 2023.

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence, 2021.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2009.

[Man22a] Lucas Mann. The 6-functor formalism for $\mathbb{Z}_\ell$- and $\mathbb{Q}_\ell$-sheaves on diamonds. `https://arxiv.org/abs/2209.08135`, 2022.

[Man22b] Lucas Mann. A p-adic 6-Functor Formalism in Rigid-Analytic Geometry. https://arxiv.org/abs/2206.02022, 2022.

[RJRC23] Joaquín Rodrigues Jacinto and Juan Esteban Rodríguez Camargo. Solid locally analytic representations, 2023.

[Sch19] Peter Scholze. Lectures on Condensed Mathematics. https://www.math.uni-bonn.de/people/scholze/Condensed.pdf, 2019.