

SINGULAR HOMOLOGY

These notes continue *Homology*, where the algebra of chain complexes used below — chain homotopies, the long exact sequence of a short exact sequence of complexes, the five lemma, tensor products, Tor and Ext — was developed. References to that companion are given as *Theorem such-and-such of Homology*; all other references are internal. All maps of spaces are continuous. We define the singular homology functor for topological spaces and pairs, show it is a homotopy functor and establish several of its basic properties.

1. SINGULAR HOMOLOGY

We carry out the first step of the process described at the beginning of §2 of *Homology*, and manufacture a chain complex out of a topological space.

1.1. The singular chain complex.

Definition 1.1. The *standard n -simplex* is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \text{ for all } i, \sum_i t_i = 1 \right\},$$

the compact convex subset of $\mathbb{R}^{n+1}$ with *vertices* the vectors $e_0, \dots, e_n$ of the standard basis. For $0 \leq i \leq n$ the *i -th face map*

$$\delta^i: \Delta^{n-1} \longrightarrow \Delta^n$$

is the affine linear map determined by $e_j \mapsto e_j$ for $j < i$ and $e_j \mapsto e_{j+1}$ for $j \geq i$; its image is the face of Δ^n opposite the vertex e_i . For $i < j$ the two ways of including Δ^{n-2} as a face of a face agree in the sense that

$$(1.1) \quad \delta^j \circ \delta^i = \delta^i \circ \delta^{j-1}: \Delta^{n-2} \longrightarrow \Delta^n,$$

as one checks on vertices.

Definition 1.2. A *singular n -simplex* in a space X is a map $\sigma: \Delta^n \rightarrow X$; the set of these is written $\text{Sing}_n(X)$. The group of *singular n -chains* is the free abelian group

$$C_n(X) = \mathbb{Z}[\text{Sing}_n(X)]$$

for $n \geq 0$, and $C_n(X) = 0$ for $n < 0$; thus a chain is a finite formal sum $\sum_k m_k \sigma_k$ with $m_k \in \mathbb{Z}$ and $\sigma_k \in \text{Sing}_n(X)$. The *boundary* $\partial_n: C_n(X) \rightarrow C_{n-1}(X)$ is the homomorphism determined on generators by

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta^i.$$

Lemma 1.3. $\partial_{n-1} \circ \partial_n = 0$, so that $(C_*(X), \partial)$ is a non-negative chain complex of abelian groups, the singular chain complex of X .

Proof. For a singular n -simplex σ ,

$$\partial\partial\sigma = \sum_{j=0}^{n-1} \sum_{i=0}^n (-1)^{i+j} \sigma \circ \delta^i \circ \delta^j.$$

Split the sum into the terms with $i \leq j$ and those with $i > j$. By (1.1), read with $j+1$ in place of j , one has $\delta^i \circ \delta^j = \delta^{j+1} \circ \delta^i$ whenever $i \leq j$; so the term indexed by such a pair (i, j) equals the term indexed by the pair $(j+1, i)$, which has first index strictly greater than its second. This is a bijection between the two families of terms, and the signs $(-1)^{i+j}$ and $(-1)^{(j+1)+i}$ are opposite. The terms therefore cancel in pairs. $\square$

Proposition 1.4. *A map $f: X \rightarrow Y$ induces a chain map $f_\#: C_*(X) \rightarrow C_*(Y)$, determined on generators by $f_\#(\sigma) = f \circ \sigma$, and $(g \circ f)_\# = g_\# \circ f_\#$, $(\text{id}_X)_\# = \text{id}$. Thus $X \mapsto C_*(X)$, $f \mapsto f_\#$, is a functor from the category of topological spaces and maps to the category of non-negative chain complexes of abelian groups and chain maps.*

Proof. Composition with f carries a singular simplex of X to one of Y , so $f_\#$ is defined; and it commutes with the boundary because composition is associative:

$$\partial f_\#(\sigma) = \sum_i (-1)^i (f \circ \sigma) \circ \delta^i = \sum_i (-1)^i f \circ (\sigma \circ \delta^i) = f_\#(\partial\sigma).$$

The two identities are clear on generators. $\square$

1.2. The singular homology.

Definition 1.5. The *singular homology* of X is $H_n(X) = H_n(C_*(X))$. For a ring R one puts $C_*(X; R) = R[\text{Sing}_*(X)]$, the free R -module on the singular simplices with the same boundary formula, and $H_n(X; R) = H_n(C_*(X; R))$; this is a complex of R -modules, so everything proved in *Homology* applies to it. We write $H_n(X)$ for $H_n(X; \mathbb{Z})$.

Proposition 1.6. *H_n is a functor from the category of topological spaces and maps to the category of abelian groups: a map $f: X \rightarrow Y$ induces $H_n(f): H_n(X) \rightarrow H_n(Y)$, and $H_n(g \circ f) = H_n(g) \circ H_n(f)$, $H_n(\text{id}_X) = \text{id}$.*

Proof. H_n is the composite of the functor of Proposition 1.4 with the homology functor of Proposition 2.8. $\square$

We denote by $H_*(X)$ the graded abelian group $\{H_n(X)\}_{n \in \mathbb{Z}}$, the *singular homology* of X . It is a functor from the category of topological spaces to the category of graded abelian groups, carrying a map $f: X \rightarrow Y$ to $H_*(f) = \{H_n(f)\}_{n \in \mathbb{Z}}$.

The properties of this invariant — its computation for simple spaces, its homotopy invariance, its reduced and relative forms, and the long exact sequence of a pair — are the subject of the sections that follow.

Definition 1.7. The *singular cohomology* of X , written $H^*(X)$, is the cohomology of the cochain complex given by

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z}), \quad \delta^n = \partial_{n+1}^*: C^n(X) \longrightarrow C^{n+1}(X),$$

the algebraic dual of $\partial_{n+1}: C_{n+1}(X) \rightarrow C_n(X)$; that is, $\delta^n \varphi = \varphi \circ \partial_{n+1}$. Then $\delta^{n+1} \delta^n = 0$, because $\partial_{n+1} \partial_{n+2} = 0$.

Proposition 1.8. *Singular cohomology is a contravariant functor from spaces to graded abelian groups: a map $f: X \rightarrow Y$ induces $f^*: H^*(Y) \rightarrow H^*(X)$, with $(g \circ f)^* = f^* \circ g^*$ and $(\text{id}_X)^* = \text{id}$.*

Proof. Write $f^\# = \text{Hom}(f_\#, \mathbb{Z})$ for the dual of the chain map $f_\#$ of Proposition 1.4. Then $f_\# \partial = \partial f_\#$ gives $\delta f^\# = f^\# \delta$; so $f^\#$ is a cochain map, and it is contravariantly functorial because $\text{Hom}(-, \mathbb{Z})$ is. It therefore induces maps on cohomology, functorially. $\square$

2. HOMOLOGY AND COHOMOLOGY WITH COEFFICIENTS

Nothing in the constructions of §1 used the integers in particular. Replacing $\mathbb{Z}$ by an abelian group M gives two more invariants, and every result proved for H_* and H^* carries over to them with the same proof; we record the statements and indicate the citations. Homotopy invariance is deferred to Corollary 4.3, where it is proved for all coefficients at once.

Definition 2.1. Let M be an abelian group. The singular chain complex of X with coefficients in M is

$$C_*(X; M) = C_*(X) \otimes_{\mathbb{Z}} M,$$

and the singular cochain complex with coefficients in M is $C^n(X; M) = \text{Hom}(C_n(X), M)$ with $\delta^n = \partial_{n+1}^*$, as in Definition 1.7. Their homology and cohomology are the *singular homology and cohomology of X with coefficients in M* , written $H_*(X; M)$ and $H^*(X; M)$. For $M = \mathbb{Z}$ they are the groups of §1. More generally, for a commutative ring R and an R -module M one puts

$$C_*(X; M) = C_*(X; R) \otimes_R M, \quad C^n(X; M) = \text{Hom}_R(C_n(X; R), M),$$

with $C_*(X; R)$ as in Definition 1.5; since that complex is free on $\text{Sing}_*(X)$, the case $R = \mathbb{Z}$ is the one above.

For an R -module M the two constructions agree, and for the same reason in both cases: $C_n(X)$ is free abelian on $\text{Sing}_n(X)$ and $C_n(X; R)$ is free as an R -module on that same set, so each of $C_n(X) \otimes_{\mathbb{Z}} M$ and $C_n(X; R) \otimes_R M$ is the direct sum of copies of M indexed by $\text{Sing}_n(X)$. Concretely, a chain of $C_n(X; M)$ is a finite sum $\sum_k m_k \sigma_k$ with $m_k \in M$ and $\sigma_k \in \text{Sing}_n(X)$, and a cochain of $C^n(X; M)$ is an arbitrary function $\text{Sing}_n(X) \rightarrow M$, the two descriptions of the cochains agreeing because an R -linear map on a free R -module and a homomorphism on a free abelian group are each determined by arbitrary values on the basis. The boundary and coboundary are given by the same formulas as before. So the underlying abelian groups coincide; what the R -version adds is the R -module structure these groups carry when M is an R -module.

Proposition 2.2. *$H_*(-; M)$ is a functor, and $H^*(-; M)$ a contravariant functor, from spaces to graded abelian groups, and to graded R -modules when M is an R -module.*

Proof. Apply $- \otimes M$ and $\text{Hom}(-, M)$ to the chain map $f_\#$ of Proposition 1.4: the first gives a chain map $f_\# \otimes \text{id}_M$, the second a cochain map $\text{Hom}(f_\#, M)$, and both constructions are functorial, the second contravariantly. The arguments of Propositions 1.6 and 1.8 then apply verbatim. $\square$

The relation between the groups with coefficients in M and those with coefficients in the ground ring is the content of the universal coefficient theorems of §8 of *Homology*, which we state here for spaces. In the two results that follow R is a principal ideal domain and M an R -module, and $H_*(X; M)$ and $H^*(X; M)$ are as in Definition 2.1. The complex $C_*(X; R)$ is free in every degree, which is the hypothesis those theorems require.

Theorem 2.3 (universal coefficients in homology). *For every space X and every R -module M there is a split short exact sequence*

$$0 \longrightarrow H_n(X; R) \otimes M \longrightarrow H_n(X; M) \longrightarrow \operatorname{Tor}(H_{n-1}(X; R), M) \longrightarrow 0.$$

Proof. Apply Theorem 8.1 of *Homology* to $C_*(X; R)$. □

Theorem 2.4 (universal coefficients in cohomology). *For every space X and every R -module M there is a split short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X; R), M) \longrightarrow H^n(X; M) \longrightarrow \operatorname{Hom}(H_n(X; R), M) \longrightarrow 0.$$

In particular, if F is a field then $H^n(X; F) \cong \operatorname{Hom}_F(H_n(X; F), F)$: with field coefficients cohomology is the dual of homology.

Proof. Apply Theorem 8.6 of *Homology* to $C_*(X; R)$. Over a field every module is free, so the Ext term vanishes by Proposition 8.4(ii). □

Remark 2.5. The Ext term shifts torsion up by one degree: if $H_{n-1}(X; \mathbb{Z})$ contains a summand $\mathbb{Z}/m$, then $H^n(X; \mathbb{Z})$ contains $\operatorname{Ext}(\mathbb{Z}/m, \mathbb{Z}) = \mathbb{Z}/m$, while $\operatorname{Hom}(\mathbb{Z}/m, \mathbb{Z}) = 0$ means that this torsion is invisible to the first term. So integral cohomology in degree n sees the free part of H_n and the torsion of H_{n-1} .

Exercise 2.6. Show that $H^0(X; M) \cong \operatorname{Hom}(H_0(X), M)$ for every X and M , and reconcile this with the description of H^0 as the group of functions $\pi_0(X) \rightarrow M$ given in *Singular Cohomology*.

Exercise 2.7. Suppose $H_*(X; R)$ is finitely generated and free in every degree. Show that $H^n(X; R) \cong \operatorname{Hom}(H_n(X; R), R)$ for every n , and that both are free of the same rank.

Exercise 2.8. Show that $H^n(X; \mathbb{Q}) \cong \operatorname{Hom}(H_n(X; \mathbb{Z}), \mathbb{Q})$ for every X , and explain why no Ext term appears.

3. FIRST COMPUTATIONS

Proposition 3.1. *Let P be a one-point space. Then $H_0(P) = \mathbb{Z}$ and $H_n(P) = 0$ for $n \neq 0$.*

Proof. For each $n \geq 0$ there is exactly one map $\Delta^n \rightarrow P$, say σ_n , so $C_n(P) = \mathbb{Z}$ generated by σ_n . We have $\partial\sigma_0 = 0$, and for $n \geq 1$, since $\sigma_n \circ \delta^i = \sigma_{n-1}$ for every i ,

$$\partial\sigma_n = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0, & n \text{ odd}, \\ \sigma_{n-1}, & n \text{ even}. \end{cases}$$

So the complex is $\cdots \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$ ending in degree 0. In degree 0, $Z_0 = \mathbb{Z}$ and $B_0 = 0$ because $\partial_1 = 0$, so $H_0 = \mathbb{Z}$. For $n \geq 1$ odd, $Z_n = \mathbb{Z}$ and $B_n = \operatorname{im} \partial_{n+1} = \mathbb{Z}$, since $n+1$ is even; for n even, $Z_n = \ker \partial_n = 0$. Either way $H_n = 0$. □

Proposition 3.2. *For any space X , $H_0(X)$ is the free abelian group on the set $\pi_0(X)$ of path components of X . In particular $H_0(X) \cong \mathbb{Z}$ when X is non-empty and path connected, generated by the class of any point.*

Proof. A singular 0-simplex is a map $\Delta^0 \rightarrow X$, that is, a point of X , so $C_0(X)$ is free abelian on the points of X , and $Z_0(X) = C_0(X)$ because $\partial_0 = 0$. A singular 1-simplex is a path $\tau: \Delta^1 \rightarrow X$, and by the definitions of δ^0, δ^1 ,

$$\partial\tau = \tau \circ \delta^0 - \tau \circ \delta^1 = \tau(e_1) - \tau(e_0),$$

the endpoint minus the starting point. Hence $B_0(X)$ is the subgroup generated by all differences $y - x$ with x and y in the same path component of X . The quotient of the free abelian group on the points of X by that subgroup is the free abelian group on $\pi_0(X)$: the surjection $C_0(X) \rightarrow \mathbb{Z}[\pi_0(X)]$ sending a point to its component kills $B_0(X)$, and any element of its kernel has, in each component, coefficient sum zero, hence is a sum of differences $y - x$ within components. $\square$

Proposition 3.3. *Let $X = \coprod_{\alpha} X_{\alpha}$ be a disjoint union of spaces. Then the inclusions induce an isomorphism of complexes $\bigoplus_{\alpha} C_*(X_{\alpha}) \cong C_*(X)$, and hence*

$$H_n(X) \cong \bigoplus_{\alpha} H_n(X_{\alpha}) \quad \text{for every } n.$$

Proof. The simplex Δ^n is connected, so the image of a singular simplex of X is connected and lies in a single X_{α} . Thus $\text{Sing}_n(X)$ is the disjoint union of the $\text{Sing}_n(X_{\alpha})$, and the free abelian group on a disjoint union is the direct sum of the free abelian groups; the identification is compatible with ∂ , since the faces of a simplex with image in X_{α} again have image there. The second statement now follows from Remark 2.13 of *Homology*. $\square$

4. HOMOTOPY INVARIANCE

We now prove the statement promised in Remark 3.9 of *Homology*: the singular chain complex converts homotopies of maps into chain homotopies. For points $p_0, \dots, p_k$ of a convex subset K of a Euclidean space we write

$$[p_0, \dots, p_k]: \Delta^k \longrightarrow K$$

for the affine linear map with $e_j \mapsto p_j$.

Theorem 4.1. *Let $f, g: X \rightarrow Y$ be homotopic maps. Then $f_{\#}$ and $g_{\#}$ are chain homotopic, and consequently $H_n(f) = H_n(g)$ for every n .*

Proof. Let $h: X \times I \rightarrow Y$ be a homotopy with $h(\cdot, 0) = f$ and $h(\cdot, 1) = g$. The prism $\Delta^n \times I$ is a convex subset of $\mathbb{R}^{n+1} \times \mathbb{R}$; write

$$v_i = (e_i, 0), \quad w_i = (e_i, 1) \quad (0 \leq i \leq n)$$

for its vertices. For a singular n -simplex σ of X put $F_{\sigma} = h \circ (\sigma \times \text{id}_I): \Delta^n \times I \rightarrow Y$ and define

$$P(\sigma) = \sum_{i=0}^n (-1)^i F_{\sigma} \circ [v_0, \dots, v_i, w_i, \dots, w_n] \in C_{n+1}(Y),$$

extended to a homomorphism $P: C_n(X) \rightarrow C_{n+1}(Y)$. Geometrically the $n + 1$ simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$ triangulate the prism. We shall show that

$$(4.1) \quad \partial P + P\partial = g_{\#} - f_{\#},$$

so that P is a chain homotopy from $g_{\#}$ to $f_{\#}$ in the sense of Definition 3.1 of *Homology*.

In the simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$ the vertex v_j ($j \leq i$) stands in position j , and the vertex w_j ($j \geq i$) stands in position $j + 1$. Omitting them in turn,

$$\begin{aligned} \partial P(\sigma) = & \sum_{j \leq i} (-1)^{i+j} F_{\sigma} \circ [v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n] \\ & + \sum_{j \geq i} (-1)^{i+j+1} F_{\sigma} \circ [v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]. \end{aligned}$$

Consider first the terms with $i \neq j$. On the other side, $P(\partial\sigma) = \sum_j (-1)^j P(\sigma \circ \delta^j)$, and $F_{\sigma \circ \delta^j}$ is F_{σ} composed with the inclusion of the j -th face of the prism; expanding P gives exactly the terms above with $j < i$ in the first sum and $j > i$ in the second, with the opposite signs. Hence the terms with $i \neq j$ in $\partial P(\sigma)$ sum to $-P(\partial\sigma)$.

There remain the terms with $i = j$. These are

$$\begin{aligned} & + F_{\sigma} \circ [v_0, \dots, v_{i-1}, w_i, \dots, w_n] && \text{from the first sum,} \\ & - F_{\sigma} \circ [v_0, \dots, v_i, w_{i+1}, \dots, w_n] && \text{from the second.} \end{aligned}$$

The i -th term of the second family cancels the $(i + 1)$ -st term of the first, and the sum telescopes to

$$F_{\sigma} \circ [w_0, \dots, w_n] - F_{\sigma} \circ [v_0, \dots, v_n] = g \circ \sigma - f \circ \sigma = g_{\#}(\sigma) - f_{\#}(\sigma),$$

since $F_{\sigma}(x, 1) = g(\sigma(x))$ and $F_{\sigma}(x, 0) = f(\sigma(x))$. Together the two computations give $\partial P(\sigma) = g_{\#}(\sigma) - f_{\#}(\sigma) - P(\partial\sigma)$, which is Equation (4.1). The last assertion is Theorem 3.3 of *Homology*. $\square$

Corollary 4.2. *Singular homology is a homotopy functor: homotopic maps induce the same homomorphism on homology, so the functor H_* factors through the homotopy category of spaces. In particular a homotopy equivalence $f: X \rightarrow Y$ induces isomorphisms $H_n(X) \rightarrow H_n(Y)$ for every n ; in fact $f_{\#}$ is a chain homotopy equivalence, and so a quasi-isomorphism.*

Proof. If $f \simeq g$ then $f_{\#} \simeq g_{\#}$ by Theorem 4.1, and chain homotopic maps induce the same map on homology; so the functor of Proposition 1.6 identifies homotopic maps and therefore factors through the homotopy category. For the rest, let g be a homotopy inverse of f . Then $g_{\#}f_{\#} = (g \circ f)_{\#}$ is chain homotopic to $(\text{id}_X)_{\#} = \text{id}$ by Theorem 4.1 and Proposition 1.4, and similarly $f_{\#}g_{\#} \simeq \text{id}$. So $f_{\#}$ is a chain homotopy equivalence, and Corollary 3.7 of *Homology* applies. $\square$

Corollary 4.3. *Let M be an abelian group. Homotopic maps $f \simeq g: X \rightarrow Y$ induce the same homomorphisms*

$$H_*(X; M) \longrightarrow H_*(Y; M), \quad H^*(Y; M) \longrightarrow H^*(X; M).$$

So homology and cohomology with coefficients in M — in particular singular cohomology, the case $M = \mathbb{Z}$ — are homotopy functors, factoring through the homotopy category of spaces, and a homotopy equivalence induces isomorphisms.

Proof. By Theorem 4.1 there is a chain homotopy P with $\partial P + P\partial = g_\# - f_\#$. Applying $- \otimes M$, the map $P \otimes \text{id}_M$ is a chain homotopy from $g_\# \otimes \text{id}_M$ to $f_\# \otimes \text{id}_M$, by Proposition 6.6 of *Homology*. Applying $\text{Hom}(-, M)$, the dual maps $\text{Hom}(P, M): C^{n+1}(Y; M) \rightarrow C^n(X; M)$ satisfy $\text{Hom}(P, M)\delta + \delta \text{Hom}(P, M) = g^\# - f^\#$, both sides carrying a cochain φ to $\varphi \circ (\partial P + P\partial)$. Chain homotopic chain maps induce the same map on homology, by Theorem 3.3 of *Homology*, and the same argument applies to cochain complexes. For the last statement, a homotopy inverse g of f has $f^\# g^\# = (g \circ f)^\#$ inducing the identity, and likewise in the other order. $\square$

Corollary 4.4. *If X is contractible then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \neq 0$. In particular this holds for any convex subset of a Euclidean space, and for $\mathbb{R}^k$.*

Proof. A contractible space is homotopy equivalent to a point, so Corollary 4.2 and Proposition 3.1 give the result. A convex set is contractible by the straight-line homotopy to any of its points. $\square$

Exercise 4.5. Show that the $(n+1)$ -simplices listed in the formula for $P(\sigma)$ exactly triangulate $\Delta^n \times I$, and that the signs compare the product orientation of $\Delta^n \times I$ with the orientation of each of these $(n+1)$ -simplices coming from the ordering of its vertices.

5. REDUCED HOMOLOGY

Definition 5.1. The *augmentation* is the homomorphism $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ with $\varepsilon(\sigma) = 1$ for every singular 0-simplex σ . Since $\varepsilon(\partial\tau) = 1 - 1 = 0$ for a singular 1-simplex τ , the *augmented singular complex*

$$\cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

with $\mathbb{Z}$ placed in degree -1 , is a chain complex; its homology is the *reduced singular homology* $\tilde{H}_*(X)$.

A map $f: X \rightarrow Y$ satisfies $\varepsilon \circ f_\# = \varepsilon$ in degree 0, since $f_\#$ carries a singular 0-simplex to a singular 0-simplex; so $f_\#$ extends to a chain map of augmented complexes, the identity on the copy of $\mathbb{Z}$ in degree -1 . Hence $\tilde{H}_*$ is again a functor from spaces to graded abelian groups, and it is a homotopy functor: the prism operator vanishes in degree -1 , so the chain homotopy of Theorem 4.1 is one of augmented complexes.

Proposition 5.2. *For non-empty X one has $\tilde{H}_n(X) = H_n(X)$ for $n \geq 1$, and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particular $\tilde{H}_n(P) = 0$ for all n when P is a point.*

Proof. Above degree 0 the two complexes agree, whence the first statement, and $\tilde{H}_n = H_n = 0$ for $n < 0$. In degree 0 the cycles of the augmented complex are $\ker \varepsilon$, while the boundaries are $B_0(X)$ as before, so $\tilde{H}_0(X) = \ker \varepsilon / B_0(X)$. Choosing a point $x \in X$ splits ε , and $C_0(X) = \ker \varepsilon \oplus \mathbb{Z}x$; passing to the quotient by $B_0(X)$, which is contained in $\ker \varepsilon$, gives $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$. For a point, ε is an isomorphism in degree 0, so $\tilde{H}_0 = 0$, and the higher groups vanish by Proposition 3.1. $\square$

Remark 5.3. With homotopy invariance in hand, two things remain before singular homology can be computed for spaces of interest. First, the long exact sequence of a pair (X, A) , obtained by applying Theorem 4.3 of *Homology* to the short exact sequence of

complexes $0 \rightarrow C_*(A) \rightarrow C_*(X) \rightarrow C_*(X)/C_*(A) \rightarrow 0$. Second, excision, which is what makes the relative groups computable. These are the subjects of the sections to come.

Exercise 5.4. Show that $H_n(X) = 0$ for all n when X is empty, and that $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$.

Exercise 5.5. Let X be a space and $x \in X$. Show that the composite $C_*(\{x\}) \rightarrow C_*(X) \rightarrow C_*(\{x\})$ induced by the inclusion and by the constant map is the identity, and deduce that $H_n(\{x\}) \rightarrow H_n(X)$ is a split injection.

Exercise 5.6. Let X be non-empty. Show that ε induces a surjection $\varepsilon_*: H_0(X) \rightarrow \mathbb{Z}$ with kernel $\tilde{H}_0(X)$, and that ε_* is an isomorphism if and only if X is path connected.

6. RELATIVE HOMOLOGY AND THE LONG EXACT SEQUENCE OF A PAIR

A *pair* (X, A) is a space X together with a subspace $A \subseteq X$; a *map of pairs* $f: (X, A) \rightarrow (Y, B)$ is a map $f: X \rightarrow Y$ with $f(A) \subseteq B$. We allow $A = \emptyset$, and write X for the pair $(X, \emptyset)$.

6.1. Relative chains. A singular simplex of A is a singular simplex of X whose image lies in A , and distinct simplices of A remain distinct in X ; so $\text{Sing}_n(A)$ is a subset of $\text{Sing}_n(X)$ and the inclusion induces an injection $C_*(A) \hookrightarrow C_*(X)$ of complexes, whose cokernel in degree n is free abelian on the singular simplices of X whose image is *not* contained in A .

Definition 6.1. The complex of *relative singular chains* of the pair (X, A) is the quotient complex

$$C_*(X, A) = C_*(X)/C_*(A),$$

with the differential induced by ∂ , and the *relative singular homology* of the pair is $H_n(X, A) = H_n(C_*(X, A))$. Thus $C_*(X, \emptyset) = C_*(X)$ and $H_n(X, \emptyset) = H_n(X)$. With coefficients in a ring R one defines $C_*(X, A; R) = C_*(X; R)/C_*(A; R)$ and $H_n(X, A; R)$ in the same way.

Pairs of spaces and maps of pairs form a category. A map of pairs $f: (X, A) \rightarrow (Y, B)$ has $f_\#(C_*(A)) \subseteq C_*(B)$, so the chain map of Proposition 1.4 descends to the quotients; consequently $(X, A) \mapsto C_*(X, A)$ and $(X, A) \mapsto H_*(X, A)$ are functors from the category of topological pairs to, respectively, chain complexes of abelian groups and graded abelian groups. The details are in Proposition 6.5.

Proposition 6.2. *For every pair (X, A) the sequence of complexes*

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

is short exact, and is split in each degree.

Proof. Exactness in each degree is the definition of the quotient together with the injectivity noted above. In degree n the group $C_n(X)$ is free abelian on $\text{Sing}_n(X)$, which is the disjoint union of $\text{Sing}_n(A)$ and its complement; the subgroup generated by the complement is a splitting of the quotient map. $\square$

Remark 6.3. Unwinding the quotient, a class in $H_n(X, A)$ is represented by a chain $c \in C_n(X)$ with $\partial c \in C_{n-1}(A)$ — a *relative cycle* — and two relative cycles c, c' represent the same class exactly when $c - c' = \partial b + a$ for some $b \in C_{n+1}(X)$ and some $a \in C_n(A)$. So the relative groups measure the chains of X that are cycles only up to A .

6.2. The long exact sequence.

Theorem 6.4. *For every pair (X, A) there is a long exact sequence*

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \cdots,$$

in which the unlabelled maps are induced by the inclusions $A \subseteq X$ and $(X, \emptyset) \subseteq (X, A)$, and the connecting homomorphism is given by $\partial[c] = [\partial c]$ for a relative cycle c . Assigning to a pair its long exact sequence is a functor from the category of topological pairs to the category of long exact sequences of abelian groups; this is Proposition 6.5.

Proof. Apply Theorem 4.3 of *Homology* to the short exact sequence of Proposition 6.2. For the description of ∂ , follow the recipe of Lemma 4.2 of *Homology*: a class in $H_n(X, A)$ is represented by a relative cycle c , which is itself a lift of its own class to $C_n(X)$; its boundary ∂c lies in $C_{n-1}(A)$, and that chain, a cycle of A , represents $\partial[c]$. $\square$

Proposition 6.5. *A map of pairs $f: (X, A) \rightarrow (Y, B)$ induces a chain map $C_*(X, A) \rightarrow C_*(Y, B)$ and homomorphisms $H_n(X, A) \rightarrow H_n(Y, B)$, functorially; and the long exact sequences of (X, A) and of (Y, B) form a commutative ladder, the square containing the connecting homomorphisms commuting as well.*

Proof. Since $f(A) \subseteq B$, the chain map $f_\#$ carries $C_*(A)$ into $C_*(B)$ and so descends to the quotients; functoriality follows from that of $f_\#$ (Proposition 1.4). The resulting diagram of short exact sequences commutes, so Proposition 4.4 of *Homology* applies. $\square$

Theorem 6.6. *Let $f, g: (X, A) \rightarrow (Y, B)$ be maps of pairs which are homotopic through maps of pairs, that is, by a homotopy $h: X \times I \rightarrow Y$ with $h(A \times I) \subseteq B$. Then f and g induce the same homomorphisms $H_n(X, A) \rightarrow H_n(Y, B)$. Consequently a homotopy equivalence of pairs induces isomorphisms on relative homology. Thus relative homology, and with it the long exact sequence of Theorem 6.4, are homotopy functors defined on the category of pairs of topological spaces.*

Proof. Let P be the prism operator of the proof of Theorem 4.1, built from h . If σ is a singular simplex with image in A then $F_\sigma = h \circ (\sigma \times \text{id}_I)$ has image in $h(A \times I) \subseteq B$, so P carries $C_n(A)$ into $C_{n+1}(B)$ and descends to a homomorphism $\bar{P}: C_n(X, A) \rightarrow C_{n+1}(Y, B)$. The identity $\partial P + P\partial = g_\# - f_\#$ descends with it, so $\bar{P}$ is a chain homotopy between the induced maps of relative complexes, and Theorem 3.3 of *Homology* applies. The statement about homotopy equivalences of pairs follows as in Corollary 4.2; and since homotopic maps of pairs induce the same map on each of $H_*(A)$, $H_*(X)$ and $H_*(X, A)$, and the ladder of Proposition 6.5 commutes, the long exact sequence too depends only on the homotopy classes. $\square$

6.3. The reduced sequence, and first consequences.

Proposition 6.7. *Let $A \neq \emptyset$. Then there is a long exact sequence*

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \longrightarrow \cdots$$

with the same relative terms.

Proof. Augment both complexes as in Definition 5.1. The inclusion is the identity on the copies of $\mathbb{Z}$ in degree -1 , so the sequence of augmented complexes

$$0 \longrightarrow \tilde{C}_*(A) \longrightarrow \tilde{C}_*(X) \longrightarrow C_*(X, A) \longrightarrow 0$$

is exact in every degree: in degree -1 it reads $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow 0$, and in degrees ≥ 0 it is Proposition 6.2. Now apply Theorem 4.3 of *Homology*. $\square$

Corollary 6.8. *For $x_0 \in X$ one has $H_n(X, \{x_0\}) \cong \tilde{H}_n(X)$ for every n .*

Proof. In the sequence of Proposition 6.7 with $A = \{x_0\}$ all the groups $\tilde{H}_n(\{x_0\})$ vanish, by Proposition 5.2. $\square$

Corollary 6.9. *If the inclusion $A \hookrightarrow X$ is a homotopy equivalence — for instance if A is a deformation retract of X — then $H_n(X, A) = 0$ for every n .*

Proof. By Corollary 4.2 the maps $H_n(A) \rightarrow H_n(X)$ are isomorphisms for all n . In the long exact sequence, $H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow H_{n-1}(X)$ has image $\ker(H_{n-1}(A) \rightarrow H_{n-1}(X)) = 0$, so it is zero; hence $H_n(X) \rightarrow H_n(X, A)$ is surjective. But its kernel is the image of the surjection $H_n(A) \rightarrow H_n(X)$, so it is also zero. Therefore $H_n(X, A) = 0$. $\square$

Remark 6.10. The long exact sequence relates the three groups of a pair, but by itself it computes nothing: one still needs a way of identifying $H_*(X, A)$ with the homology of something smaller. That is the role of excision, which says that a subspace whose closure lies in the interior of A may be cut out of both X and A without changing the relative groups. It is proved by the subdivision technique of §7, and is the subject of §8.

Exercise 6.11. Let $B \subseteq A \subseteq X$. Show that $0 \rightarrow C_*(A, B) \rightarrow C_*(X, B) \rightarrow C_*(X, A) \rightarrow 0$ is a short exact sequence of complexes, and deduce the long exact sequence of the triple

$$\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow \cdots$$

Check that for $B = \emptyset$ it is the sequence of Theorem 6.4.

Exercise 6.12. Show that $H_n(X, A) \cong \bigoplus_{\alpha} H_n(X_{\alpha}, A \cap X_{\alpha})$, where the X_{α} are the path components of X .

Exercise 6.13. Let $A \subseteq X$ be non-empty. Show that $H_0(X, A)$ is free abelian on the path components of X that do not meet A .

7. THE MAYER-VIETORIS THEOREM

If $X = A \cup B$ one would like to compute $H_*(X)$ from $H_*(A)$, $H_*(B)$ and $H_*(A \cap B)$. The theorem of this section says that this can be done whenever the two interiors already cover: if $X = \text{int } A \cup \text{int } B$, where int denotes the interior, then there is a long exact sequence

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots,$$

in which the first map is $x \mapsto (x, -x)$ and the second $(a, b) \mapsto a + b$, both induced by inclusions.

The obstacle to proving it is that a singular simplex of X need not lie in A or in B : it may straddle the two, and there is no reason for the chains that do lie in one of them to carry all the homology. The remedy is to subdivide, and the content of the section is that

subdividing changes nothing at the level of homology. Once that is known the theorem is three lines.

We prove it once for a pair of spaces, Theorem 7.19, taking the sequence above as the special case $Y = \emptyset$, Corollary 7.20. The relative form costs only the bookkeeping of four subspaces, and it is what is needed for the Mayer–Vietoris sequence of a pair of CW complexes and for products of pairs.

7.1. Affine chains and barycentric subdivision. Throughout this section K denotes a non-empty convex subset of a Euclidean space. For $p_0, \dots, p_n \in K$ we write $[p_0, \dots, p_n]: \Delta^n \rightarrow K$ for the affine linear map with $e_j \mapsto p_j$, as in §4; these are the *affine simplices* of K .

Definition 7.1. Let $LC_n(K) \subseteq C_n(K)$ be the subgroup generated by the affine n -simplices, for $n \geq 0$. It is closed under taking boundaries and hence is a subcomplex of $C_*(K)$. Let $LC_{-1}(K) = \mathbb{Z}$, generated by a symbol $[\]$ called the *empty simplex*. Extend the boundary by $\partial[p_0] = [\]$. The identity $\partial\partial = 0$ is inherited from $C_*(K)$ except in degree 1, where $\partial\partial[p_0, p_1] = \partial([p_1] - [p_0]) = [\] - [\] = 0$; so $LC_*(K)$ is a chain complex, the *augmented affine complex* of K .

Definition 7.2. For $b \in K$ the *cone* operator $b \cdot : LC_n(K) \rightarrow LC_{n+1}(K)$ is defined on generators by

$$b \cdot [p_0, \dots, p_n] = [b, p_0, \dots, p_n], \quad b \cdot [\] = [b].$$

Lemma 7.3. $\partial(b \cdot \lambda) + b \cdot (\partial\lambda) = \lambda$ for every $\lambda \in LC_*(K)$. Thus the cone operator is a chain contraction, and $LC_*(K)$ is exact.

Proof. For $\lambda = [p_0, \dots, p_n]$ the first face of $[b, p_0, \dots, p_n]$ omits b and the remaining ones are the cones on the faces of λ :

$$\partial[b, p_0, \dots, p_n] = [p_0, \dots, p_n] - \sum_i (-1)^i [b, p_0, \dots, \widehat{p_i}, \dots, p_n] = \lambda - b \cdot \partial\lambda.$$

For $\lambda = [\]$ we have $\partial(b \cdot [\]) = \partial[b] = [\] = \lambda$ and $b \cdot \partial[\] = 0$. The identity says $\partial \circ (b \cdot) + (b \cdot) \circ \partial = \text{id}$, so id is null-homotopic and the homology of $LC_*(K)$ vanishes. $\square$

Definition 7.4. The *barycenter* of $\lambda = [p_0, \dots, p_n]$ is $b_\lambda = \frac{1}{n+1} \sum_i p_i \in K$. The *barycentric subdivision* operator $S: LC_n(K) \rightarrow LC_n(K)$ is defined inductively by

$$S[\] = [\], \quad S\lambda = b_\lambda \cdot (S\partial\lambda).$$

The operator $T: LC_n(K) \rightarrow LC_{n+1}(K)$ is defined inductively by

$$T[\] = 0, \quad T\lambda = b_\lambda \cdot (\lambda - T\partial\lambda).$$

In degree 0, $S[p_0] = p_0 \cdot [\] = [p_0]$, so S is the identity on LC_0 .

Lemma 7.5. On $LC_*(K)$ one has

$$\partial S = S\partial, \quad \partial T + T\partial = \text{id} - S.$$

Moreover S and T are natural for affine linear maps: if $f: K \rightarrow L$ is the restriction of an affine linear map of Euclidean spaces then $f_\# S = S f_\#$ and $f_\# T = T f_\#$.

Proof. All four assertions are proved by induction on the degree, the case of $[\]$ being trivial. Assuming $\partial S = S\partial$ in degrees below that of λ , Lemma 7.3 gives

$$\partial S\lambda = \partial(b_\lambda \cdot S\partial\lambda) = S\partial\lambda - b_\lambda \cdot \partial S\partial\lambda = S\partial\lambda - b_\lambda \cdot S\partial\partial\lambda = S\partial\lambda.$$

For the second identity, assume it in lower degrees; applied to $\partial\lambda$ it gives

$$\partial T\partial\lambda = \partial\lambda - S\partial\lambda - T\partial\partial\lambda = \partial\lambda - S\partial\lambda.$$

Hence, by Lemma 7.3 again,

$$\begin{aligned} \partial T\lambda &= \partial(b_\lambda \cdot (\lambda - T\partial\lambda)) = (\lambda - T\partial\lambda) - b_\lambda \cdot (\partial\lambda - \partial T\partial\lambda) \\ &= \lambda - T\partial\lambda - b_\lambda \cdot S\partial\lambda = \lambda - T\partial\lambda - S\lambda. \end{aligned}$$

Finally an affine linear f carries $[p_0, \dots, p_n]$ to $[fp_0, \dots, fp_n]$ and therefore carries barycenters to barycenters, $f(b_\lambda) = b_{f\#\lambda}$; the naturality of S and of T follows from the inductive definitions. $\square$

Lemma 7.6. *Let $\lambda = [p_0, \dots, p_n]$.*

- (i) *The diameter of the image of λ is $\max_{i,j} |p_i - p_j|$.*
- (ii) *Every affine simplex occurring in $S\lambda$ has diameter at most $\frac{n}{n+1} \text{diam } \lambda$.*

Consequently every simplex occurring in $S^m\lambda$ has diameter at most $\left(\frac{n}{n+1}\right)^m \text{diam } \lambda$, and these tend to 0 as $m \rightarrow \infty$.

Proof. (i) It is clear that $\max_{i,j} |p_i - p_j|$ is at most the diameter of the image of λ . For any $u \in K$ the distance function $v \mapsto |u - v|$ is convex, and it is constant on no non-degenerate interval, since a segment is not contained in a Euclidean sphere; so on any non-degenerate interval $I \subseteq K$ it takes its maximum value only at one (or at both) of the endpoints. Hence the point of λ furthest from u must be one of the p_i , and thus $\max_{v \in \lambda} |u - v| = \max_i |u - p_i|$. If in addition $u \in \lambda$, the same argument applied with u replaced by p_i and v replaced by u shows that $|p_i - u| \leq \max_j |p_i - p_j|$. It follows that the diameter of λ is at most $\max_{i,j} |p_i - p_j|$, proving (i).

(ii) First, for the barycenter $b = b_\lambda$,

$$|b - p_j| = \left| \frac{1}{n+1} \sum_i (p_i - p_j) \right| \leq \frac{1}{n+1} \sum_{i \neq j} |p_i - p_j| \leq \frac{n}{n+1} \text{diam } \lambda,$$

and hence, by the computation in (i), $|b - v| \leq \frac{n}{n+1} \text{diam } \lambda$ for every point v of the image of λ . Now argue by induction on n , the case $n = 0$ being trivial. Every simplex of $S\lambda$ is of the form $[b, q_0, \dots, q_{n-1}]$ with $[q_0, \dots, q_{n-1}]$ a simplex of $S\mu$ for some face μ of λ . By (i) its diameter is the largest distance between two of its vertices. Two vertices among the q 's are at distance at most $\frac{n-1}{n} \text{diam } \mu \leq \frac{n}{n+1} \text{diam } \lambda$ by induction, and a distance $|b - q_k|$ is at most $\frac{n}{n+1} \text{diam } \lambda$ by the estimate just proved, proving (ii). The last statement follows by iterating, the simplices of $S^m\lambda$ being simplices of S applied to those of $S^{m-1}\lambda$, all of dimension n . $\square$

7.2. Subdivision of singular chains. Let $\iota_n = [e_0, \dots, e_n] \in LC_n(\Delta^n)$ denote the identity map of Δ^n regarded as an affine simplex.

Definition 7.7. For a space X define homomorphisms

$$S: C_n(X) \rightarrow C_n(X), \quad T: C_n(X) \rightarrow C_{n+1}(X)$$

on generators by $S\sigma = \sigma_\#(S\iota_n)$ and $T\sigma = \sigma_\#(T\iota_n)$.

Proposition 7.8. *The operators S and T are natural: $f_\#S = Sf_\#$ and $f_\#T = Tf_\#$ for every map $f: X \rightarrow Y$. They satisfy*

$$\partial S = S\partial, \quad \partial T + T\partial = \text{id} - S,$$

and every simplex occurring in $S\sigma$ or in $T\sigma$ has image contained in the image of σ . In particular, if σ has image in a subspace $A \subseteq X$ then so do all the simplices of $S\sigma$ and $T\sigma$.

Proof. Naturality is immediate: $f_\#S\sigma = f_\#\sigma_\#(S\iota_n) = (f\sigma)_\#(S\iota_n) = S(f_\#\sigma)$, and likewise for T . The last statement holds because every simplex occurring in $\sigma_\#(\mu)$ is σ composed with an affine linear map $\Delta^k \rightarrow \Delta^n$.

For the identities, use Lemma 7.5 and the naturality there. Each δ^i is affine linear and $\partial\iota_n = \sum_i (-1)^i (\delta^i)_\# \iota_{n-1}$, so

$$\partial S\sigma = \sigma_\# \partial S\iota_n = \sigma_\# S \partial\iota_n = \sum_i (-1)^i \sigma_\# (\delta^i)_\# S\iota_{n-1} = \sum_i (-1)^i (\sigma \circ \delta^i)_\# S\iota_{n-1} = S\partial\sigma.$$

The same computation with Lemma 7.5 gives

$$\partial T\sigma = \sigma_\# \partial T\iota_n = \sigma_\# (\iota_n - S\iota_n - T\partial\iota_n) = \sigma - S\sigma - T\partial\sigma.$$

(In degree 0 the term $T\partial\iota_0 = T[\]$ vanishes, in agreement with $\partial\sigma = 0$ in the unaugmented complex $C_*(X)$.) $\square$

Definition 7.9. For $m \geq 0$ put $D_m = \sum_{i=0}^{m-1} TS^i: C_n(X) \rightarrow C_{n+1}(X)$, so that $D_0 = 0$.

Lemma 7.10. $\partial D_m + D_m\partial = \text{id} - S^m$.

Proof. Using $S^i\partial = \partial S^i$ and Proposition 7.8,

$$\partial D_m + D_m\partial = \sum_{i=0}^{m-1} (\partial TS^i + T\partial S^i) = \sum_{i=0}^{m-1} (\text{id} - S)S^i = \text{id} - S^m,$$

the sum telescoping. $\square$

Exercise 7.11. Compute $S\iota_1$ and $S\iota_2$ explicitly, and check that they are the classical barycentric subdivisions of the interval and of the triangle, each simplex occurring once and with the sign determined by the ordering of its vertices.

7.3. Chains that are small for a cover.

Definition 7.12. Let $\mathcal{U} = \{U_j\}$ be a family of subspaces of X whose interiors cover X . A singular simplex is $\mathcal{U}$ -small if its image lies in some U_j , and $C_n^{\mathcal{U}}(X)$ denotes the subgroup of $C_n(X)$ generated by the $\mathcal{U}$ -small simplices. The faces of a $\mathcal{U}$ -small simplex are $\mathcal{U}$ -small, so $C_*^{\mathcal{U}}(X)$ is a subcomplex, and by Proposition 7.8 the operators S and T carry it into itself.

Theorem 7.13. *Let $\kappa: C_*^{\mathcal{U}}(X) \rightarrow C_*(X)$ be the inclusion. There are a chain map $\rho: C_*(X) \rightarrow C_*^{\mathcal{U}}(X)$ and homomorphisms $D: C_n(X) \rightarrow C_{n+1}(X)$ with*

$$\rho\kappa = \text{id}, \quad \partial D + D\partial = \text{id} - \kappa\rho, \quad D(C_*^{\mathcal{U}}(X)) = 0.$$

In particular κ is a chain homotopy equivalence, and $H_n(C_^{\mathcal{U}}(X)) \rightarrow H_n(X)$ is an isomorphism for every n .*

Proof. The number $m(\sigma)$. Let σ be a singular n -simplex. The sets $\sigma^{-1}(\text{int } U_j)$ form an open cover of the compact metric space Δ^n , so by the Lebesgue covering lemma there is an $\eta > 0$ such that every subset of Δ^n of diameter less than η is carried by σ into some U_j . By Lemma 7.6 the simplices of $S^m \iota_n$ have diameter less than η for m large, and then $S^m \sigma = \sigma_{\#}(S^m \iota_n)$ lies in $C_n^{\mathcal{U}}(X)$. Let $m(\sigma)$ be the smallest such m . If $\sigma_j = \sigma \circ \delta^j$ is a face, then $m(\sigma_j) \leq m(\sigma)$: the subdivision of a face of Δ^n is the corresponding face of the subdivision, so every simplex of $S^m \sigma_j$ has image contained in that of some simplex of $S^m \sigma$.

Construction. Define D on generators by $D\sigma = D_{m(\sigma)}\sigma$ and put

$$\rho = \text{id} - \partial D - D\partial.$$

Then ρ is a chain map, since $\partial\rho = \partial - \partial D\partial = \rho\partial$, and $\text{id} - \rho = \partial D + D\partial$ by construction. If σ is $\mathcal{U}$ -small then $m(\sigma) = 0$ and $m(\sigma_j) = 0$ for all faces, so $D\sigma = 0$ and $D\partial\sigma = 0$, whence D vanishes on $C_*^{\mathcal{U}}(X)$ and $\rho\sigma = \sigma$ there; this gives $\rho\kappa = \text{id}$.

ρ lands in $C^{\mathcal{U}}$. By Lemma 7.10, $\sigma - \partial D_{m(\sigma)}\sigma = S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma$, so

$$\rho(\sigma) = S^{m(\sigma)}\sigma + D_{m(\sigma)}\partial\sigma - D\partial\sigma = S^{m(\sigma)}\sigma + \sum_j (-1)^j (D_{m(\sigma)} - D_{m(\sigma_j)})(\sigma_j).$$

The first term lies in $C_*^{\mathcal{U}}(X)$ by the definition of $m(\sigma)$. Since $m(\sigma_j) \leq m(\sigma)$, the j -th term is $(D_{m(\sigma)} - D_{m(\sigma_j)})(\sigma_j) = \sum_{i=m(\sigma_j)}^{m(\sigma)-1} TS^i(\sigma_j)$, and for $i \geq m(\sigma_j)$ the chain $S^i\sigma_j$ is $\mathcal{U}$ -small, hence so is $TS^i\sigma_j$. Therefore $\rho(\sigma) \in C_*^{\mathcal{U}}(X)$.

Thus $\rho\kappa = \text{id}$ and $\kappa\rho \simeq \text{id}$, so κ is a chain homotopy equivalence and induces isomorphisms on homology. $\square$

7.4. Excisive couples.

Definition 7.14. An *excisive couple* in a space X is a pair A, B of subspaces with $X = \text{int } A \cup \text{int } B$, the interiors being taken in X . We write $(X; A, B)$ for such a couple and

$$C_*^{A,B}(X) = C_*(A) + C_*(B) \subseteq C_*(X)$$

for the subcomplex of chains that are sums of a chain of A and a chain of B ; it is the complex $C_*^{\mathcal{U}}(X)$ of Definition 7.12 for the cover $\mathcal{U} = \{A, B\}$.

Definition 7.15. A pair (X, Y) is *excisive for the couple* $(X; A, B)$ if $(X; A, B)$ is an excisive couple, $Y \subseteq X$, and $(Y; A \cap Y, B \cap Y)$ is an excisive couple in Y . We then write

$$A_Y = A \cap Y, \quad B_Y = B \cap Y, \quad \text{so that} \quad A_Y \cap B_Y = A \cap B \cap Y.$$

The case $Y = \emptyset$ is allowed and is the absolute case.

Remark 7.16. The condition on Y is automatic when Y is open in X : then $\text{int}_Y(A \cap Y) \supseteq (\text{int } A) \cap Y$ and likewise for B , so $Y = ((\text{int } A) \cap Y) \cup ((\text{int } B) \cap Y)$ is covered by the two interiors. For general Y it is a hypothesis.

7.5. The chain-level sequence. Throughout, (X, Y) is excisive for $(X; A, B)$, and $A_Y = A \cap Y$ and $B_Y = B \cap Y$. Abbreviate

$$C_*^{A,B}(X, Y) = C_*^{A,B}(X) / C_*^{A_Y, B_Y}(Y),$$

which makes sense because $C_*(A_Y) \subseteq C_*(A)$ and $C_*(B_Y) \subseteq C_*(B)$, so that $C_*^{A_Y, B_Y}(Y)$ is a subcomplex of $C_*^{A,B}(X)$.

Lemma 7.17. *The maps $x \mapsto (x, -x)$ and $(a, b) \mapsto a + b$ give a short exact sequence of complexes*

$$0 \longrightarrow C_*(A \cap B, A_Y \cap B_Y) \longrightarrow C_*(A, A_Y) \oplus C_*(B, B_Y) \longrightarrow C_*^{A,B}(X, Y) \longrightarrow 0.$$

Proof. All four groups are quotients of subgroups of $C_*(X)$, and we use throughout that a subspace $Z \subseteq X$ has $C_*(Z)$ spanned by the subset $\text{Sing}_*(Z)$ of the basis $\text{Sing}_*(X)$; hence $C_*(Z) \cap C_*(Z') = C_*(Z \cap Z')$ for subspaces Z, Z' .

Injectivity. Let $x \in C_*(A \cap B)$ have $(x, -x) = 0$, that is $x \in C_*(A_Y)$ and $x \in C_*(B_Y)$. Then $x \in C_*(A_Y \cap B_Y)$, so x is zero in $C_*(A \cap B, A_Y \cap B_Y)$.

Surjectivity. A chain of $C_*^{A,B}(X)$ is $a + b$ with $a \in C_*(A)$ and $b \in C_*(B)$, which is the image of (a, b) .

Exactness in the middle. The composite is zero, since $x - x = 0$. Conversely suppose $a \in C_*(A)$ and $b \in C_*(B)$ have $a + b \in C_*^{A_Y, B_Y}(Y)$, say $a + b = c + d$ with $c \in C_*(A_Y)$ and $d \in C_*(B_Y)$. Then

$$a - c = d - b \in C_*(A) \cap C_*(B) = C_*(A \cap B),$$

because $c \in C_*(A_Y) \subseteq C_*(A)$ and $d \in C_*(B_Y) \subseteq C_*(B)$. Call this chain x . Then $(a, b) - (x, -x) = (c, d)$ lies in $C_*(A_Y) \oplus C_*(B_Y)$, so (a, b) and $(x, -x)$ have the same class in $C_*(A, A_Y) \oplus C_*(B, B_Y)$. $\square$

Lemma 7.18. *The inclusion $C_*^{A,B}(X, Y) \rightarrow C_*(X, Y)$ is a quasi-isomorphism.*

Proof. By Theorem 7.13, applied to the cover $\{A, B\}$ of X and to the cover $\{A_Y, B_Y\}$ of Y , the inclusions $C_*^{A,B}(X) \rightarrow C_*(X)$ and $C_*^{A_Y, B_Y}(Y) \rightarrow C_*(Y)$ are chain homotopy equivalences, in particular quasi-isomorphisms. Compare the two short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{A_Y, B_Y}(Y) & \longrightarrow & C_*^{A,B}(X) & \longrightarrow & C_*^{A,B}(X, Y) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_*(Y) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, Y) \longrightarrow 0 \end{array}$$

with vertical maps the inclusions. The rows are exact by definition of the quotients, the diagram commutes, and the first two vertical maps are quasi-isomorphisms; by the five lemma (Exercise 4.6 of *Homology*) applied to the ladder of long exact sequences of Theorem 4.3, the third is one as well. $\square$

7.6. The theorem.

Theorem 7.19 (Mayer–Vietoris for pairs). *Let (X, Y) be excisive for $(X; A, B)$. There is a long exact sequence*

$$\begin{aligned} \cdots &\longrightarrow H_n(A \cap B, A_Y \cap B_Y) \longrightarrow H_n(A, A_Y) \oplus H_n(B, B_Y) \\ &\longrightarrow H_n(X, Y) \longrightarrow H_{n-1}(A \cap B, A_Y \cap B_Y) \longrightarrow \cdots, \end{aligned}$$

in which the first map is $x \mapsto (x, -x)$ and the second $(a, b) \mapsto a + b$, both induced by inclusions of pairs. It is natural: a map of spaces carrying one set of such data into another produces a commutative diagram between the two long exact sequences.

Proof. Apply Theorem 4.3 of *Homology* to the short exact sequence of Lemma 7.17 and replace $H_n(C_*^{A,B}(X, Y))$ by $H_n(X, Y)$ using Lemma 7.18. For naturality, a map f carrying $(X, Y; A, B)$ into $(X', Y'; A', B')$ carries each of the four complexes of Lemma 7.17 into its primed counterpart and commutes with the two maps of that lemma; so it gives a map of short exact sequences, and Proposition 4.4 of *Homology* applies. The identification of Lemma 7.18 is induced by an inclusion, hence natural as well. $\square$

Corollary 7.20 (the absolute sequence). *Let $A, B \subseteq X$ satisfy $X = \text{int } A \cup \text{int } B$. There is a long exact sequence*

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots,$$

with the maps of Theorem 7.19. If $A \cap B \neq \emptyset$ there is a corresponding sequence in reduced homology.

Proof. Take $Y = \emptyset$ in Theorem 7.19; the relative groups of the pairs $(A, \emptyset)$ and so on are the absolute ones. For the reduced form, augment the four complexes of Lemma 7.17; in degree -1 the sequence reads $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$, which is exact. $\square$

Corollary 7.21 (relative Mayer–Vietoris). *Let $X = \text{int } A \cup \text{int } B$ and let $Y \subseteq X$ be open. Then there is a long exact sequence*

$$\cdots \longrightarrow H_n(A \cap B, A \cap B \cap Y) \longrightarrow H_n(A, A \cap Y) \oplus H_n(B, B \cap Y) \longrightarrow H_n(X, Y) \longrightarrow \cdots.$$

Proof. An open Y is excisive for $(X; A, B)$ by Remark 7.16, so this is Theorem 7.19 with the notation unwound. $\square$

8. EXCISION

Theorem 8.1 (excision). *Let $A, B \subseteq X$ satisfy $X = \text{int } A \cup \text{int } B$. Then the inclusion of pairs $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A) \quad \text{for every } n.$$

Proof. Take $\mathcal{U} = \{A, B\}$, so that $C_*^{\mathcal{U}}(X) = C_*(A) + C_*(B)$ inside $C_*(X)$.

A singular simplex has image in A and in B precisely when it has image in $A \cap B$, and these subgroups are generated by subsets of the basis $\text{Sing}_*(X)$; hence $C_*(A) \cap C_*(B) = C_*(A \cap B)$. The composite $C_*(B) \rightarrow C_*^{\mathcal{U}}(X) \rightarrow C_*^{\mathcal{U}}(X)/C_*(A)$ is therefore surjective with kernel $C_*(A \cap B)$, and induces an isomorphism of complexes

$$C_*(B)/C_*(A \cap B) \xrightarrow{\cong} C_*^{\mathcal{U}}(X)/C_*(A).$$

Next compare the two short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*^{\mathcal{U}}(X) & \longrightarrow & C_*^{\mathcal{U}}(X)/C_*(A) \longrightarrow 0 \\ & & \parallel & & \downarrow \kappa & & \downarrow \\ 0 & \longrightarrow & C_*(A) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, A) \longrightarrow 0. \end{array}$$

The left vertical map is the identity and the middle one is a quasi-isomorphism by Theorem 7.13; by the five lemma (Exercise 4.6 of *Homology*) applied to the ladder of long exact sequences, the right-hand map is a quasi-isomorphism as well. Combining the two displays gives the assertion, and the isomorphism is the one induced by the inclusion of pairs. $\square$

Corollary 8.2. *Let $Z \subseteq A \subseteq X$ with $\overline{Z} \subseteq \text{int } A$. Then the inclusion $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms*

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A) \quad \text{for every } n.$$

Proof. Put $B = X \setminus Z$. Then $\text{int } B \supseteq X \setminus \overline{Z}$, so $\text{int } A \cup \text{int } B \supseteq \text{int } A \cup (X \setminus \overline{Z}) = X$ because $\overline{Z} \subseteq \text{int } A$; and $A \cap B = A \setminus Z$. Apply Theorem 8.1. $\square$