

SIMPLICIAL COMPLEXES

Simplicial complexes are the combinatorial models of spaces: a space is described by a list of vertices and a rule saying which finite sets of them span a simplex. Everything is finite data, so everything is computable; the price is that a given space has many models and that maps of spaces are only approximated by maps of models. These notes set up the objects, the maps, the resulting category, the one-dimensional examples — graphs, forests, trees — and the subdivision that makes the approximation possible. We use the vocabulary of categories, functors and full subcategories from the companion notes.

1. ABSTRACT SIMPLICIAL COMPLEXES

Definition 1.1. An *abstract simplicial complex* consists of a set V of *vertices* together with a collection K of finite non-empty subsets of V , called *simplices*, such that

- (K1) $\{v\} \in K$ for every $v \in V$; and
- (K2) if $\sigma \in K$ and $\emptyset \neq \tau \subseteq \sigma$ then $\tau \in K$.

We denote the complex by the letter K of its collection of simplices and write $V(K) = V$ for its vertex set; by (K1) the collection determines the vertex set, so nothing is lost.

A *simplex* is a simplicial complex that contains all the non-empty subsets of its vertices. It follows that the vertex set of a simplex is finite. One says the simplex is the simplex *on* its vertices or *generated* by its vertices. A simplex with $n + 1$ vertices is an *n -simplex*, and n is its *dimension*. The non-empty subsets $\tau \subseteq \sigma$ of a simplex σ are its *faces*.

For any simplicial complex K , each $\sigma \in K$ is a simplex in its own right (since all non-empty subsets of its vertices are simplices in K). The *dimension* of K is $\dim K = \sup\{\dim \sigma : \sigma \in K\} \in \{0, 1, 2, \dots\} \cup \{\infty\}$. The complex is *finite* if $V(K)$ is finite, or equivalently if K has only finitely many simplices.

Condition (K1) says that we do not carry vertices that span no simplex; (K2) is the essential axiom, that a simplex comes with all of its faces. We freely identify a vertex v with the 0-simplex $\{v\}$.

Definition 1.2. A *subcomplex* of K is a subset $L \subseteq K$ that is itself a simplicial complex, i.e. is closed under passage to non-empty faces. The *n -skeleton* $K^{(n)}$ is the subcomplex of simplices of dimension $\leq n$.

Example 1.3 (Simplices and their boundaries). We write $\Delta(S)$ for the simplex on a finite non-empty set S ; writing $\Delta^n = \Delta(\{0, 1, \dots, n\})$ we get the standard n -simplex as a combinatorial object. Its *boundary* $\partial\Delta^n = \Delta^n \setminus \{\{0, \dots, n\}\}$, obtained by deleting the top simplex, is a subcomplex; it is the $(n - 1)$ -skeleton of Δ^n .

2. GEOMETRIC REALIZATION

A simplicial complex is a piece of combinatorics. This section attaches to it a topological space, in such a way that a simplex on $n + 1$ vertices becomes an honest geometric n -simplex — the convex hull of $n + 1$ affinely independent points.

Definition 2.1 (Barycentric coordinates). Let K be a simplicial complex. Write $\mathbb{R}^{V(K)}$ for the real vector space of finitely supported functions $V(K) \rightarrow \mathbb{R}$, with standard basis $\{e_v\}_{v \in V(K)}$, so that $t = \sum_v t(v)e_v$. The *support* of t is $\text{supp } t = \{v : t(v) \neq 0\}$. The *geometric realization* of K is

$$|K| = \left\{ t \in \mathbb{R}^{V(K)} : t(v) \geq 0 \text{ for all } v, \quad \sum_v t(v) = 1, \quad \text{supp } t \in K \right\}.$$

The numbers $t(v)$ are the *barycentric coordinates* of the point t . For $\sigma \in K$ put

$$|\sigma| = \{t \in |K| : \text{supp } t \subseteq \sigma\},$$

the *closed simplex* of σ . The condition $\sum_v t(v) = 1$ already forces $\text{supp } t$ to be non-empty and finite, so the requirement $\text{supp } t \in K$ is the only combinatorial input: it says that the vertices actually used by t span a simplex of K .

If $V(K)$ is finite, all topologies are clear. $\mathbb{R}^{V(K)}$ is a finite-dimensional $\mathbb{R}$ -vector space and has its usual topology. It is easy to see (Exercise 6.1) that $|K| \subset \mathbb{R}^{V(K)}$ is a closed subset — it is the union of the finitely many closed simplices $|\sigma|$, each of them compact. It has the induced topology. When $V(K)$ is infinite, we need to be a little more careful. In this case, the space $\mathbb{R}^{V(K)}$ is given the weak topology, meaning that a subset of $\mathbb{R}^{V(K)}$ is closed if and only if its intersection with every finite-dimensional linear subspace is closed. Since every finite-dimensional linear subspace is contained in $\mathbb{R}^{V'}$ for some finite subset $V' \subset V(K)$, it is equivalent to say that $X \subset \mathbb{R}^{V(K)}$ is closed if and only if for every such V' the intersection of X with $\mathbb{R}^{V'}$ is closed. This in turn is testing X against the finite subcomplexes of K , as the proof of Lemma 2.4 makes precise. Then $|K| \subset \mathbb{R}^{V(K)}$ is given the subspace topology of the weak topology on $\mathbb{R}^{V(K)}$.

We begin with the simplest case, the geometric realization of a simplex.

Proposition 2.2 (the realization of a simplex). *Let $V = \{v_0, \dots, v_n\}$ be a finite set and let σ be the simplex it generates. Let δ_i be the characteristic function of v_i , so that $\{\delta_i\}_{i=0}^n$ is the standard basis of $\mathbb{R}^V$. Then $|\sigma|$ is the convex hull in $\mathbb{R}^V$ of these basis elements. In particular $|\sigma|$ is homeomorphic to the n -disk D^n . The barycentric coordinates on $|\sigma|$ assign to a function $t : V \rightarrow \mathbb{R}$ its values at $v_0, \dots, v_n$.*

Proof. Every non-empty subset of V is a simplex of σ , so for $K = \sigma$ the support condition of Definition 2.1 is automatic and

$$|\sigma| = \left\{ \sum_{i=0}^n t_i \delta_i : t_i \geq 0, \quad \sum_{i=0}^n t_i = 1 \right\} = \text{conv}\{\delta_0, \dots, \delta_n\} \subseteq \mathbb{R}^V,$$

which is the asserted convex hull.

The δ_i form a basis of $\mathbb{R}^V$ and are in particular affinely independent: if $\sum_i a_i \delta_i = 0$ with $\sum_i a_i = 0$, then already $a_i = 0$ for every i . Hence a point of $|\sigma|$ is a convex combination of the δ_i in exactly one way, and its coefficients are $t_i = t(v_i)$; these are the barycentric coordinates.

Finally $|\sigma|$ is closed and bounded in $\mathbb{R}^V$, hence compact, and it is a convex subset of the n -dimensional affine subspace $A = \{x \in \mathbb{R}^V : \sum_i x_i = 1\}$ with non-empty interior in A — the barycenter $\frac{1}{n+1} \sum_i \delta_i$ is such an interior point. A compact convex subset of an n -dimensional affine space with non-empty interior is homeomorphic to D^n . $\square$

Thus for a simplex $\sigma \in K$ the closed simplex $|\sigma|$ of Definition 2.1 is a geometric n -simplex, $n = \dim \sigma$, sitting inside $|K|$ via $\mathbb{R}^\sigma \subseteq \mathbb{R}^{V(K)}$, and the faces of σ give exactly the faces of that geometric simplex. The realization is assembled from these pieces along common faces, and from nothing else:

Lemma 2.3. $|K| = \bigcup_{\sigma \in K} |\sigma|$, and for $\sigma, \tau \in K$

$$|\sigma| \cap |\tau| = \begin{cases} |\sigma \cap \tau|, & \sigma \cap \tau \neq \emptyset, \\ \emptyset, & \sigma \cap \tau = \emptyset. \end{cases}$$

Proof. A point $t \in |K|$ lies in $|\sigma|$ for $\sigma = \text{supp } t$, which is a simplex by definition; this gives the first assertion. For the second, $t \in |\sigma| \cap |\tau|$ means $\text{supp } t \subseteq \sigma$ and $\text{supp } t \subseteq \tau$, i.e. $\text{supp } t \subseteq \sigma \cap \tau$, and $\text{supp } t \neq \emptyset$. $\square$

Each $|\sigma|$ carries the Euclidean topology of Proposition 2.2, which is the topology it inherits from $|K|$, since $\mathbb{R}^\sigma$ is a finite-dimensional subspace of $\mathbb{R}^{V(K)}$. The topology of $|K|$ is determined by these pieces:

Lemma 2.4. For $A \subseteq |K|$ the following are equivalent:

- (i) A is closed in $|K|$;
- (ii) $A \cap |\sigma|$ is closed for every simplex $\sigma \in K$;
- (iii) $A \cap |K'|$ is closed for every finite subcomplex $K' \subset K$.

Proof. For a finite subset $V' \subset V(K)$ put $K_{V'} = \{\sigma \in K : \sigma \subseteq V'\}$, a finite subcomplex, and note that

$$|K| \cap \mathbb{R}^{V'} = \{t \in |K| : \text{supp } t \subseteq V'\} = \bigcup_{\sigma \in K_{V'}} |\sigma| = |K_{V'}|,$$

a finite union of compact simplices and hence a closed subset of $\mathbb{R}^{V'}$. Since $\mathbb{R}^{V(K)}$ is given the weak topology from the subspaces $\mathbb{R}^{V'}$ for V' a finite subset of $V(K)$, and $|K|$ carries the induced topology, (i) and (iii) are equivalent: testing A against $\mathbb{R}^{V'}$ is testing it against $|K_{V'}|$, and every finite subcomplex is contained in some $K_{V'}$.

(ii) and (iii) are equivalent since if X is a finite union of closed subsets X_i then $Y \subset X$ is closed if and only if $Y \cap X_i$ is closed for every i (Exercise 6.2): apply this to $X = |K'| = \bigcup_{\sigma \in K'} |\sigma|$ for a finite subcomplex K' , and note that every simplex of K lies in the finite subcomplex it generates. $\square$

Thus the topology of $|K|$ is the weak topology determined by its closed simplices, or equivalently by its finite subcomplexes: a set is closed as soon as it is closed on each finite piece.

Definition 2.5. A space homeomorphic to some $|K|$ is a *polyhedron*, and such a K is a *triangulation* of it.

Example 2.6. $|\Delta^n|$ is the standard n -simplex, and $|\partial \Delta^n|$ is its boundary sphere, so $|\partial \Delta^n| \cong S^{n-1}$.

Definition 2.7 (Barycenter). The *barycenter* of a simplex σ is the point

$$\hat{\sigma} = \frac{1}{\dim \sigma + 1} \sum_{v \in \sigma} \delta_v \in |\sigma|,$$

the unique point of $|\sigma|$ with all barycentric coordinates equal. It lies in the interior of $|\sigma|$ relative to the affine subspace it spans.

3. SIMPLICIAL MAPS AND THE CATEGORY **SC**

Definition 3.1. Let K and L be simplicial complexes. A *simplicial map* $f: K \rightarrow L$ is a function $f: V(K) \rightarrow V(L)$ such that

$$\sigma \in K \implies f(\sigma) = \{f(v) : v \in \sigma\} \in L.$$

A simplicial map is thus determined by what it does to vertices, and the only requirement is that simplices go to simplices. Note that f need not be injective on a simplex: the image $f(\sigma)$ may have smaller dimension than σ , and one says that f *collapses* σ . This is not a defect but a necessity — without collapsing there would be no simplicial map from a subdivided interval to a point.

Proposition 3.2. *Simplicial complexes and simplicial maps form a category **SC**.*

Proof. If $f: K \rightarrow L$ and $g: L \rightarrow M$ are simplicial then so is the composite of the two vertex functions, since $(g \circ f)(\sigma) = g(f(\sigma))$ is a simplex of M whenever $\sigma \in K$; composition of functions is associative. The identity function of $V(K)$ is simplicial and is a two-sided unit. $\square$

Definition 3.3. A simplicial map $f: K \rightarrow L$ is an *isomorphism* in **SC** if it has a two-sided inverse simplicial map, equivalently if $f: V(K) \rightarrow V(L)$ is a bijection and $\sigma \in K \Leftrightarrow f(\sigma) \in L$.

The two conditions really are needed: the identity map of $\{0, 1, 2\}$, regarded as a simplicial map $\partial\Delta^2 \rightarrow \Delta^2$, is a bijection on vertices, but its inverse is not simplicial, since the top simplex of Δ^2 is not a simplex of $\partial\Delta^2$. This is the same phenomenon as a continuous bijection that is not a homeomorphism.

Example 3.4 (Realization is a functor). The geometric realization of §2 extends to a functor $|\cdot|: \mathbf{SC} \rightarrow \mathbf{Top}$: a simplicial map $f: K \rightarrow L$ induces $|f|: |K| \rightarrow |L|$ by

$$|f|(t)(w) = \sum_{f(v)=w} t(v),$$

that is, by extending f affinely over each simplex — the formula sends δ_v to $\delta_{f(v)}$ and is the restriction of a linear map $\mathbb{R}^{V(K)} \rightarrow \mathbb{R}^{V(L)}$, so it is affine, hence continuous, on each $|\sigma|$, hence continuous for the weak topology. It lands in $|L|$ because $\text{supp } |f|(t) = f(\text{supp } t) \in L$. Functoriality is immediate from the formula.

Example 3.5 (Ordered complexes and posets). Let **Poset** be the category of partially ordered sets and order-preserving maps. The *order complex* $\mathcal{O}(P)$ of a poset P has $V = P$ and as simplices the finite non-empty chains $p_0 < p_1 < \cdots < p_n$. An order-preserving map $\varphi: P \rightarrow Q$ carries a chain to a chain — possibly with repetitions, which is harmless because Definition 3.1 asks only that the image *set* be a simplex — so $\mathcal{O}: \mathbf{Poset} \rightarrow \mathbf{SC}$ is a functor. It is used in §4.

4. BARYCENTRIC SUBDIVISION

A simplicial map $|K| \rightarrow |L|$ is a very rigid thing, and a continuous map $|K| \rightarrow |L|$ will usually not be one. To make the simplicial maps better approximate continuous ones we refine K , keeping $|K|$ unchanged. The systematic refinement is the barycentric subdivision, and it is nothing but the order complex of the poset of simplices.

Definition 4.1. The *face poset* $\mathcal{F}(K)$ of a simplicial complex K is the set K of simplices, partially ordered by inclusion. A simplicial map $f: K \rightarrow L$ induces the order-preserving map $\mathcal{F}(f): \sigma \mapsto f(\sigma)$, so $\mathcal{F}: \mathbf{SC} \rightarrow \mathbf{Poset}$ is a functor.

Definition 4.2. The *barycentric subdivision* of K is the order complex of the poset $\mathcal{F}(K)$:

$$\mathrm{Sd} K = \mathcal{O}(\mathcal{F}(K));$$

its vertices are the simplices of K , and its n -simplices are the chains

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n, \quad \sigma_i \in K.$$

Unwinding, (K1) and (K2) hold because a subset of a chain is a chain, so $\mathrm{Sd} K$ is indeed a simplicial complex. Geometrically the vertex σ is to be thought of as the barycenter $\hat{\sigma} \in |\sigma|$ of Definition 2.7, and a chain spans the simplex with those barycenters as vertices.

Proposition 4.3. $\mathrm{Sd} = \mathcal{O} \circ \mathcal{F}: \mathbf{SC} \rightarrow \mathbf{SC}$ is a functor, $\dim \mathrm{Sd} K = \dim K$, and $\mathrm{Sd} K$ is finite whenever K is.

Proof. The first claim is Definition 4.1 together with Example 3.5: a simplicial $f: K \rightarrow L$ gives $\mathrm{Sd} f: \mathrm{Sd} K \rightarrow \mathrm{Sd} L$, $\sigma \mapsto f(\sigma)$ on vertices, and a chain $\sigma_0 \subsetneq \cdots \subsetneq \sigma_n$ maps to the set $\{f(\sigma_0), \dots, f(\sigma_n)\}$, a chain in $\mathcal{F}(L)$ — possibly shorter, since f may collapse — and functoriality is that of $\sigma \mapsto f(\sigma)$. A chain in $\mathcal{F}(K)$ has at most $\dim K + 1$ terms, since dimensions strictly increase along it, and a chain of faces of a top-dimensional simplex realizes that length. Finally the vertices of $\mathrm{Sd} K$ are the simplices of K , so $\mathrm{Sd} K$ is finite when K is. $\square$

Example 4.4 (The first two subdivisions). $\mathrm{Sd} \Delta^1$ has three vertices a, b, ab and two edges $\{a, ab\}, \{b, ab\}$: the interval subdivided at its midpoint, i.e. a path with two edges. $\mathrm{Sd} \Delta^2$ has $3 + 3 + 1 = 7$ vertices — the three vertices, the three edges and the face — and its 2-simplices are the maximal chains $v \subsetneq e \subsetneq f$: there are $3 \cdot 2 \cdot 1 = 6$ of them. This is the familiar picture of a triangle cut into six small triangles by its medians. In general the n -simplices of $\mathrm{Sd} \Delta^n$ correspond to the maximal chains in the poset of faces, i.e. to the $(n + 1)!$ orderings of the vertices.

Remark 4.5 (Subdivision keeps the space). There is a homeomorphism $h_K: |\mathrm{Sd} K| \rightarrow |K|$, defined by sending the vertex σ of $\mathrm{Sd} K$ to the barycenter $\hat{\sigma} \in |\sigma|$ and extending affinely over each simplex. So K and $\mathrm{Sd} K$ triangulate the same polyhedron; they are almost never isomorphic in $\mathbf{SC}$. It is worth noticing that the maps h_K do *not* constitute a natural transformation $|\mathrm{Sd}(-)| \Rightarrow |-|$: for the map $f: \Delta^2 \rightarrow \Delta^1$ collapsing 0 and 1 to 0 and sending 2 to 1, $|f| \circ h_{\Delta^2}$ carries the vertex of $\mathrm{Sd} \Delta^2$ given by the 2-simplex itself to the point with weights $(\frac{2}{3}, \frac{1}{3})$, whereas $h_{\Delta^1} \circ |\mathrm{Sd} f|$ carries it to the barycenter $(\frac{1}{2}, \frac{1}{2})$ of the image simplex. Naturality does hold when restricted to the simplicial maps that are injective on each simplex.

Definition 4.6. The *mesh* of a finite complex K is the largest Euclidean diameter of a closed simplex $|\sigma| \subseteq \mathbb{R}^{V(K)}$, $\sigma \in K$, in the realization of §2.

Write $\text{Sd}^m = \text{Sd} \circ \cdots \circ \text{Sd}$ for the m -fold iterate, and let $h^{(m)}: |\text{Sd}^m K| \rightarrow |K|$ be the composite of the homeomorphisms of Remark 4.5. The mesh of $\text{Sd}^m K$ *measured in* $|K|$ is the largest Euclidean diameter of a set $h^{(m)}(|\tau|) \subseteq \mathbb{R}^{V(K)}$, $\tau \in \text{Sd}^m K$. This, and not the intrinsic mesh, is the quantity that matters: measured in its own realization every simplex of $\text{Sd}^m K$ is regular with all edges of length $\sqrt{2}$, so the intrinsic mesh carries no information about the subdivision.

Lemma 4.7. *Let K be a finite complex of dimension n . Then the mesh of $\text{Sd}^m K$ measured in $|K|$ is at most*

$$\left(\frac{n}{n+1}\right)^m \text{mesh}(K) \xrightarrow{m \rightarrow \infty} 0.$$

The proof is Exercise 6.3.

This estimate is the whole point of the construction. In the lectures we shall prove the Simplicial Approximation Theorem: for any continuous $f: |K| \rightarrow |L|$ there are an $m \geq 0$ depending on f and a simplicial $g: \text{Sd}^m K \rightarrow L$ with $|g|$ homotopic to f , once $|\text{Sd}^m K|$ is identified with $|K|$ as above.

5. GRAPHS, FORESTS AND TREES

Let us consider simplicial complexes of dimension at most one; i.e., graphs.

Definition 5.1. A *graph* is a simplicial complex of dimension ≤ 1 . Its 0-simplices are its *vertices* and its 1-simplices its *edges*. We write **Graph** $\subseteq$ **SC** for the full subcategory whose objects are the graphs: no morphisms are discarded, so a morphism of graphs is a simplicial map, i.e. a map on vertices carrying each edge to an edge or a vertex.

So a graph here is what is elsewhere called a *simple* graph: an edge is a two-element set, so there are no loops and no two distinct edges with the same endpoints, and multigraphs are not simplicial complexes.

Definition 5.2. Let G be a graph. A *walk* from v to w is a finite sequence $v = v_0, \dots, v_n = w$ of vertices with $\{v_{i-1}, v_i\}$ an edge for $1 \leq i \leq n$; it is a *path* if the v_i are distinct. G is *connected* if it is non-empty and any two of its vertices are joined by a walk. A *cycle* is a sequence of *distinct* vertices $v_0, \dots, v_{n-1}$, $n \geq 3$, with $\{v_i, v_{i+1}\}$ an edge for all i , indices modulo n .

The condition $n \geq 3$ is forced by the simplicial setting: v_0, v_1, v_0 is not a cycle, since an edge is a set and traversing it twice encloses nothing.

Definition 5.3. A *forest* is a graph containing no cycle. A *tree* is a connected forest. We write **Forest** and **Tree** for the corresponding full subcategories of **Graph**, hence of **SC**.

Proposition 5.4. *For a non-empty graph T the following are equivalent:*

- (i) T is a tree;
- (ii) any two vertices of T are joined by a unique path;
- (iii) T is connected, and removing any edge disconnects it.

Moreover a finite forest with n vertices and c components has $n - c$ edges; a tree has $n - 1$.

Sketch. (i) $\Rightarrow$ (ii): existence is connectedness; two distinct paths with the same endpoints contain a cycle. (ii) $\Rightarrow$ (iii): if removing $\{v, w\}$ left v, w joined by a path, then that path and the edge itself would be two distinct paths from v to w . (iii) $\Rightarrow$ (i): a cycle through an edge would keep its endpoints joined after the edge is removed. The count follows by induction: a finite tree with at least two vertices has a vertex of degree 1, the end of a longest path, and deleting it removes one vertex and one edge; for a forest, add up the counts of its components. $\square$

Example 5.5 (Small graphs). The *path* P_n has vertices $0, 1, \dots, n$ and edges $\{i-1, i\}$; it is a tree, and $|P_n| \cong [0, 1]$. The *cycle* C_n , $n \geq 3$, has vertices $\mathbb{Z}/n$ and edges $\{i, i+1\}$; it is not a forest, and $|C_n| \cong S^1$. Note $C_3 = \partial\Delta^2$. A forest is exactly a disjoint union of trees.

Example 5.6 (Trees are not preserved by simplicial maps). Define $f: P_3 \rightarrow C_3$ on vertices by $0, 1, 2, 3 \mapsto 0, 1, 2, 0$. Each edge of P_3 goes to an edge of C_3 , so f is a simplicial map; its source is a tree and its image is all of C_3 , which is not. So the classes of trees and of forests are not closed under simplicial images: being a tree is a property of an object, not something inherited along maps. So although **Tree** and **Forest** are full subcategories of **Graph**, their classes of objects are not closed under simplicial images, whereas that of **Graph** is, since $\dim f(K) \leq \dim K$ for any simplicial f .

6. EXERCISES

Exercise 6.1. Prove that for finite K the set $|K|$ is closed in $\mathbb{R}^{V(K)}$.

Exercise 6.2. Let X be the union of finitely many closed subsets $X_1, \dots, X_k$. Prove that $Y \subset X$ is closed if and only if every $Y \cap X_i$ is, and give an example showing this fails for infinite unions.

Exercise 6.3. Prove Lemma 4.7.

Exercise 6.4. Show that a simplicial map $f: K \rightarrow L$ is determined by $|f|$, and that $|\cdot|$ is a faithful functor **SC** $\rightarrow$ **Top**. Is it full?

Exercise 6.5. Prove that a graph G is connected if and only if $|G|$ is path-connected, and that the connected components of G correspond to the path components of $|G|$.

Exercise 6.6. Show that a finite graph with n vertices and more than $n-1$ edges contains a cycle, and deduce the edge count of Proposition 5.4 from this.

Exercise 6.7. Show that every finite connected graph contains a tree with the same vertex set (a *spanning tree*). Is the assignment $G \mapsto \text{spanning tree}$ functorial?

Exercise 6.8. Describe $\text{Sd } G$ for a graph G with n vertices and e edges: how many vertices and edges does it have? Check that $\text{Sd } C_3 \cong C_6$ and that Sd of a tree is a tree.