

PSEUDOMANIFOLDS

These notes continue *Homology*, whose section on oriented simplicial homology supplies the oriented simplicial chain complex $C_*^{\text{or}}(K)$ used throughout, *Singular Homology* and *Further Properties of Singular Homology*, with the latter shortened to *Further Properties*. A pseudomanifold is a finite simplicial complex that looks like a compact manifold as far as top-dimensional counting is concerned: it is built out of n -simplices glued in pairs along a subset of their codimension-one faces. Triangulated compact manifolds are pseudomanifolds, but so are spaces with singularities, and that is the point of the notion: the little that is required is already enough to produce a fundamental homology class. The reason for introducing them is that they are the simplest geometric representatives for every homology class.

We shall use these objects, and the homology classes they carry, in *The Hurewicz theorems*, which follows these notes. The Hurewicz theorems compare homotopy groups with homology groups, and geometric representatives for the latter are essential in their proofs. References to the companions are given as *Theorem such-and-such of Homology*, and similarly for the others; all other references are internal.

We use the notation of *Homology*: K is a finite abstract simplicial complex with vertex set $V(K)$, its simplices are certain non-empty subsets of $V(K)$ closed under passage to non-empty subsets, a simplex with $p + 1$ vertices is a p -simplex, and $|K|$ is the geometric realization, in which the closed simplex $|s|$ of a p -simplex s is a convex subset of a Euclidean space homeomorphic to Δ^p . For points $p_0, \dots, p_k$ of such a closed simplex $|s|$, or more generally of any convex subset C of a Euclidean space, we write

$$\langle p_0, \dots, p_k \rangle: \Delta^k \longrightarrow C$$

for the affine singular simplex with $e_j \mapsto p_j$; this is the map written $[p_0, \dots, p_k]$ in §4 of *Singular Homology*, and the change of brackets is to separate it from the orientation $[v_0, \dots, v_p]$ of Definition 9.1 of *Homology*.

1. PSEUDOMANIFOLDS WITH BOUNDARY

Definition 1.1. A finite simplicial complex K is an n -dimensional pseudomanifold with boundary if

- (i) every simplex of K is a face of an n -simplex of K ,
- (ii) every $(n - 1)$ -simplex of K is a face of one or two n -simplices of K .

An $(n - 1)$ -simplex that is a face of exactly one n -simplex is called *free*.

Definition 1.2. The boundary ∂K of an n -dimensional pseudomanifold with boundary is the subcomplex consisting of the free $(n - 1)$ -simplices of K together with all of their faces. The pseudomanifold is *closed* if $\partial K = \emptyset$, that is, if every $(n - 1)$ -simplex is a face of exactly two n -simplices.

Remark 1.3. Condition (i) says that K has dimension n and is *pure*: there are no simplices left over that fail to lie in a top-dimensional one. Condition (ii) weakens the counting

condition of a closed manifold to the one satisfied by a compact manifold with boundary. Two further conditions are often imposed, strong connectedness and normality; we give each a name below and invoke it only where it is needed. Everything from §2 on is written for pseudomanifolds with boundary, and where the closed case is meant the word *closed* is said. For $n = 0$ there are no $(n - 1)$ -simplices, so every 0-dimensional pseudomanifold with boundary is closed, and is a finite non-empty discrete set.

Remark 1.4. The $(n - 1)$ -simplices of ∂K are precisely the free ones, since a face of dimension $n - 1$ of a free $(n - 1)$ -simplex is that simplex itself; and every simplex of ∂K is by construction a face of a free one. So if $\partial K \neq \emptyset$ it is pure of dimension $n - 1$, and it satisfies condition (i) of Definition 1.1 in dimension $n - 1$. What has to be examined is condition (ii).

Example 1.5. The complex Δ^n of all non-empty faces of the standard n -simplex is an n -dimensional pseudomanifold with boundary: its only n -simplex is Δ^n itself, and each of the $n + 1$ faces $\partial_i \Delta^n$ is free. Its boundary is the complex $\partial \Delta^n$ of all proper non-empty faces, which by Example 1.6 below is a closed $(n - 1)$ -dimensional pseudomanifold.

Example 1.6. Let $K = \partial \Delta^{n+1}$ be the complex of all proper non-empty subsets of the vertex set $\{0, 1, \dots, n+1\}$, so that $|K| \cong S^n$. Its n -simplices are the $n+2$ subsets obtained by omitting one vertex, and each $(n - 1)$ -simplex, obtained by omitting two, is a face of exactly two of them. Then K is a closed n -dimensional pseudomanifold.

Example 1.7 (non-examples). Three triangles glued along a common edge satisfy (i) of Definition 1.1 but not (ii); that edge lies in three 2-simplices. A triangle with an extra edge attached at one vertex satisfies (ii) but not (i); the extra edge lies in no 2-simplex.

The arguments of §2 onwards are run with chains of n -simplices, and Theorem 1.14 below turns on a condition satisfied by the links. Here are the two names.

Definition 1.8. Let K be a finite simplicial complex. A *chain* from an n -simplex s to an n -simplex s' is a finite sequence

$$s = s_0, s_1, \dots, s_r = s'$$

of n -simplices of K in which s_{m-1} and s_m have a common $(n - 1)$ -face for $1 \leq m \leq r$. The complex K is *strongly connected* if any two of its n -simplices are joined by such a chain.

Definition 1.9. The *link* of a simplex s of K is the subcomplex

$$\text{lk}(s) = \{u \in K : u \cap s = \emptyset \text{ and } u \cup s \in K\}.$$

Definition 1.10. An n -dimensional pseudomanifold with boundary K is *normal* if the link of every simplex of K of dimension at most $n - 2$ is connected. The condition is vacuous for $n \leq 1$.

Remark 1.11. Neither of these conditions implies the other. Strong connectedness does not give normality: the pinched disk of Example 1.19 is strongly connected and not normal, and so, in the closed case, is the complex got by triangulating S^2 so that two vertices p and q span no edge and have no common neighbour and then identifying them. Normality does not give strong connectedness, saying nothing about connectedness at all:

two disjoint copies of $\partial\Delta^3$ are normal, and no chain joins a triangle of one to a triangle of the other. Neither does connectedness of $|K|$: two copies of $\partial\Delta^3$ with a vertex of one identified to a vertex of the other have connected realization and admit no such chain, and that complex is not normal, the link of the identified vertex being a disjoint union of two circles. What is true is that normality implies strong connectedness for pseudomanifolds with connected realization (Theorem 1.16).

Remark 1.12. Definition 1.1 is the weakest of the definitions in use. Most treatments build strong connectedness into it, so that a pseudomanifold there is a complex satisfying (i) and (ii) any two of whose n -simplices are joined by a chain; others build in normality instead.

Lemma 1.13. *Let $n \geq 2$, let K be an n -dimensional pseudomanifold with boundary, and let t be an $(n-2)$ -simplex of K . Then $\text{lk}(t)$ is a graph: its vertices are the $(n-1)$ -simplices of K containing t , its edges are the n -simplices of K containing t , and it has no simplices of higher dimension. Every vertex has degree 1 or 2, and it has degree 1 exactly when the corresponding $(n-1)$ -simplex is free. Consequently $\text{lk}(t)$ is a disjoint union of finitely many paths and cycles, and the number of free $(n-1)$ -simplices of K containing t is twice the number of its path components; in particular that number is even.*

Proof. A simplex u of $\text{lk}(t)$ with $k+1$ vertices gives a simplex $u \cup t$ of K with $n+k$ vertices, so $k \leq 1$ and $\text{lk}(t)$ is at most 1-dimensional; and $u \mapsto u \cup t$ is a bijection from the vertices and edges of $\text{lk}(t)$ to the $(n-1)$ - and n -simplices of K containing t , inverse to $\sigma \mapsto \sigma \setminus t$. Under it the incidence of a vertex with an edge of $\text{lk}(t)$ corresponds to the incidence of an $(n-1)$ -simplex with an n -simplex, so the degree of a vertex is the number of n -simplices of which the corresponding $(n-1)$ -simplex is a face; by Definition 1.1(ii) that is 1 or 2, and it is 1 exactly for a free simplex.

A finite graph all of whose vertices have degree 1 or 2 is a disjoint union of paths and cycles, no component being a single vertex because no degree is 0. A cycle has no vertex of degree 1 and a path has exactly two, whence the count. $\square$

Theorem 1.14. *Let $n \geq 1$ and let K be a normal n -dimensional pseudomanifold with boundary. Then ∂K is either empty or a closed $(n-1)$ -dimensional pseudomanifold.*

Proof. Assume $\partial K \neq \emptyset$. Condition (i) for ∂K in dimension $n-1$ is Remark 1.4. For $n=1$ there are no $(n-2)$ -simplices, condition (ii) is vacuous, and ∂K is a non-empty finite discrete set, a closed 0-dimensional pseudomanifold.

Let $n \geq 2$ and let t be an $(n-2)$ -simplex of ∂K ; by Remark 1.4 it is a face of some free $(n-1)$ -simplex. By Lemma 1.13 the graph $\text{lk}(t)$ has a vertex of degree 1, and it is connected by normality; a connected finite graph with all degrees 1 or 2 and at least one vertex of degree 1 is a path, which has exactly two vertices of degree 1. So exactly two free $(n-1)$ -simplices of K contain t , that is, t is a face of exactly two $(n-1)$ -simplices of ∂K . As the $(n-2)$ -simplices of ∂K are its codimension-one simplices, this is condition (ii), and ∂K is closed because a free $(n-2)$ -simplex of ∂K would be one lying in exactly one $(n-1)$ -simplex of ∂K . $\square$

Proposition 1.15. *Let K be a normal n -dimensional pseudomanifold with boundary and let s be a k -simplex of K with $k \leq n-1$. Then $\text{lk}(s)$ is a normal $(n-k-1)$ -dimensional pseudomanifold with boundary.*

Proof. We use throughout that for a simplex u of $\text{lk}(s)$

$$(1.1) \quad \text{lk}_{\text{lk}(s)}(u) = \text{lk}_K(s \cup u),$$

both sides consisting of the simplices w of K with $w \cap (s \cup u) = \emptyset$ and $w \cup u \cup s \in K$.

(i) Let u be a simplex of $\text{lk}(s)$. Then $u \cup s$ is a simplex of K , so by Definition 1.1(i) it is a face of an n -simplex σ of K . Now $\sigma \setminus s$ is a simplex of $\text{lk}(s)$ of dimension $n - k - 1$ containing u . In particular $\text{lk}(s)$ is non-empty of dimension exactly $n - k - 1$, no simplex of $\text{lk}(s)$ having more than $n - k$ vertices.

(ii) Let u be an $(n - k - 2)$ -simplex of $\text{lk}(s)$. Then $u \cup s$ is an $(n - 1)$ -simplex of K , and $w \mapsto w \cup s$ is a bijection from the $(n - k - 1)$ -simplices of $\text{lk}(s)$ containing u to the n -simplices of K containing $u \cup s$, with inverse $\sigma \mapsto \sigma \setminus s$. By Definition 1.1(ii) there are one or two of the latter.

Normality: let u be a simplex of $\text{lk}(s)$ of dimension at most $(n - k - 1) - 2$. Then $s \cup u$ has dimension $k + \dim u + 1 \leq n - 2$, so $\text{lk}_K(s \cup u)$ is connected by the normality of K , and by (1.1) this is $\text{lk}_{\text{lk}(s)}(u)$. $\square$

Theorem 1.16. *Let $n \geq 1$ and let K be a normal n -dimensional pseudomanifold with boundary whose realization $|K|$ is connected. Then K is strongly connected.*

Proof. Induction on n .

Let $n = 1$. Then K is a finite graph in which every vertex lies in an edge, and $|K|$ is connected, so K is a connected graph. Any two of its vertices are joined by an edge path, and consecutive edges of an edge path share a vertex, that is, a 0-face; joining an endpoint of a given edge to an endpoint of another therefore produces a chain between them.

Let $n \geq 2$ and assume the theorem in dimensions $1, \dots, n - 1$. Call two n -simplices of K equivalent when a chain joins them, let $\mathcal{C}$ be one equivalence class, and put

$$Y = \bigcup_{s \in \mathcal{C}} |s|, \quad Y' = \bigcup_{s \notin \mathcal{C}} |s|,$$

the unions being over the n -simplices of K lying in and outside $\mathcal{C}$. Each is a finite union of closed simplices and so is closed, and $Y \cup Y' = |K|$ by Definition 1.1(i). Suppose $Y' \neq \emptyset$. As $|K|$ is connected and $Y \neq \emptyset$, the two closed sets meet; choose $x \in Y \cap Y'$, let t be the unique simplex of K whose open simplex contains x , and choose n -simplices $s \in \mathcal{C}$ and $s' \notin \mathcal{C}$ with $t \subseteq s$ and $t \subseteq s'$. Since $s \neq s'$ we have $\dim t \leq n - 1$.

If $\dim t = n - 1$ then s and s' are distinct n -simplices with the common $(n - 1)$ -face t , so s, s' is a chain and $s' \in \mathcal{C}$, a contradiction.

So $\dim t = k$ with $k \leq n - 2$. By Proposition 1.15 the complex $\text{lk}(t)$ is a normal pseudomanifold with boundary of dimension $n - k - 1$, and $1 \leq n - k - 1 \leq n - 1$; it is connected by the normality of K . By the inductive hypothesis $\text{lk}(t)$ is strongly connected. As in the proof of Proposition 1.15, $w \mapsto w \cup t$ is a bijection from the $(n - k - 1)$ -simplices of $\text{lk}(t)$ to the n -simplices of K containing t , and it carries a common $(n - k - 2)$ -face of two of them to a common $(n - 1)$ -face of their images. A chain in $\text{lk}(t)$ from $s \setminus t$ to $s' \setminus t$ therefore yields a chain in K from s to s' , and again $s' \in \mathcal{C}$, a contradiction.

Hence $Y' = \emptyset$: every n -simplex of K lies in $\mathcal{C}$, and K is strongly connected. $\square$

Remark 1.17. Conversely a strongly connected K satisfying Definition 1.1(i) has $|K|$ connected, being the union of the connected sets $|s|$ over a family of n -simplices any two

of which are joined by a chain of overlapping members. So for *normal* pseudomanifolds with boundary strong connectedness and connectedness of the realization are the same condition; without normality the first is the stronger, by Remark 1.11. Neither is needed for the construction of the fundamental class in §§2–3; the one place the condition is used is Proposition 2.10, whose hypothesis may therefore be read either way. We state it there in the form in which the proof uses it.

Example 1.18 (the bowtie). Normality cannot be dropped from Theorem 1.14. Let K consist of the two 2-simplices $\{v, a, b\}$ and $\{v, c, d\}$ and their faces. Every simplex is a face of a 2-simplex, and each of the six edges lies in exactly one of them, so K is a 2-dimensional pseudomanifold with boundary and all six edges are free. Hence ∂K is the whole 1-skeleton, in which the vertex v is a face of the four edges va, vb, vc, vd ; so ∂K is not a 1-dimensional pseudomanifold. Correspondingly K is not normal: $\text{lk}(v)$ is the disjoint union of the edges ab and cd .

Example 1.19 (the pinched disk). Strong connectedness is no substitute for normality in Theorem 1.14. Triangulate a disk so that two of its boundary vertices p and q have no common neighbour and span no edge, and identify p with q ; the result is a 2-dimensional pseudomanifold with boundary, and it is strongly connected because the disk is. At the image v of p and q there are four free edges, two coming from each of p and q , so again ∂K fails to be a pseudomanifold; and $\text{lk}(v)$, a disjoint union of two arcs, is disconnected.

Example 1.20 (the cone on an annulus). Normality is not inherited by boundaries, so it could not have been made part of Definition 1.1. Let L be a triangulated annulus and let $K = v * L$ be the cone on it, a 3-dimensional complex. It is a pseudomanifold with boundary: it is pure, a triangle of L lies in the single 3-simplex $v * \sigma$, and a triangle $v * e$ lies in one or two according as the edge e of L is a boundary edge or an interior one. It is normal, the links being $\text{lk}(v) = L$, $\text{lk}(w) = v * \text{lk}_L(w)$ and $\text{lk}(e) = v * \text{lk}_L(e)$ for a vertex w and an edge e of L , and $\text{lk}(vw) = \text{lk}_L(w)$, each of them connected. Its boundary is $\partial K = L \cup (v * \partial L)$, the free triangles being those of L together with the $v * e$ for e in ∂L . By Theorem 1.14 that is a closed 2-dimensional pseudomanifold, as one checks directly; but it is not normal, since $\text{lk}_{\partial K}(v) = \partial L$ is a disjoint union of two circles.

Remark 1.21. Without normality, Lemma 1.13 still gives that every $(n-2)$ -simplex of ∂K lies in an *even* number of $(n-1)$ -simplices of ∂K . That is what an orientation argument needs, and it is the reason the constructions below go through for ∂K even when K is not normal and ∂K fails condition (ii) in dimension $n-1$, as it does in Examples 1.18 and 1.19 and so is not a pseudomanifold at all: the chain ∂z_K is a cycle carried by ∂K whatever the links do.

2. FUNDAMENTAL CLASS IN ORIENTED SIMPLICIAL HOMOLOGY

This section and the next are written for a pseudomanifold with boundary K ; the closed case is the case $\partial K = \emptyset$, in which every relative group that appears reduces to an absolute one. The fundamental class lives in $H_n(|K|, |\partial K|)$, and its image under the connecting homomorphism is the fundamental class of ∂K .

Recall from Definition 9.1 of *Homology* that an *orientation* of a p -simplex s is an equivalence class of orderings of its vertices, two orderings being equivalent when they

differ by an even permutation, that $[v_0, \dots, v_p]$ denotes the orientation of the class of $(v_0, \dots, v_p)$, and that there are exactly two orientations when $p \geq 1$. Recall also the oriented chain complex $C_*^{\text{or}}(K)$ of Definition 9.2 of *Homology*: it is free on any choice of one orientation for each simplex, by Lemma 9.3, and its boundary is

$$\partial[v_0, \dots, v_p] = \sum_{i=0}^p (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_p].$$

Remark 2.1. This combinatorial notion of orientation agrees with the local homological one, and we shall pass between them without comment. Let s be an n -simplex and x a point of its open simplex. An ordering $(v_0, \dots, v_n)$ of the vertices of s gives the affine singular simplex $\langle v_0, \dots, v_n \rangle$, a relative cycle of $(|s|, |\partial s|)$ whose local class at x generates $H_n(|s|, |s| \setminus \{x\}) \cong \mathbb{Z}$; this is Lemma 3.3 below. An even permutation of the ordering leaves that class unchanged and an odd one reverses it, so what the class really depends on is the *orientation* $[v_0, \dots, v_n]$, and the two orientations of s correspond to the two generators. The sign is Lemma 2.3 of *Further Properties*: reordering the vertices by a permutation π replaces the affine simplex by its composite with the affine automorphism of Δ^n permuting the vertices, whose determinant has sign $\text{sgn } \pi$, and an orientation-reversing affine embedding sends $[\iota_n]$ to the negative of the local class.

Definition 2.2. Let ω be an orientation of a p -simplex s and let t be a codimension-one face of s . Writing $\omega = [v_0, \dots, v_p]$ and $t = s \setminus \{v_i\}$, the *orientation induced on t by ω* is

$$\partial_t \omega = (-1)^i [v_0, \dots, \widehat{v}_i, \dots, v_p] \in C_{p-1}^{\text{or}}(K),$$

a generator of the infinite cyclic subgroup of $C_{p-1}^{\text{or}}(K)$ belonging to t . That $\partial_t \omega$ does not depend on the ordering chosen to represent ω is the well-definedness of the boundary in Lemma 9.3, and by construction $\partial \omega = \sum_t \partial_t \omega$, the sum over the codimension-one faces t of s . Two orientations of t are *opposite* if their sum is zero.

Definition 2.3. Let K be an n -dimensional pseudomanifold with boundary. An *orientation* of K is a choice of orientation ω_s of each n -simplex s such that these cancel along every $(n-1)$ -simplex that is not free: for such a t , with s and s' the two n -simplices of which t is a face,

$$\partial_t \omega_s + \partial_t \omega_{s'} = 0.$$

No condition is imposed at a free $(n-1)$ -simplex, which lies in a single n -simplex. A pseudomanifold with boundary so equipped is an *oriented* one, and one admitting such a choice is *orientable*. Replacing every ω_s by $-\omega_s$ again gives an orientation, the *opposite* one. When K is closed there are no free $(n-1)$ -simplices and we speak of an *oriented closed n -dimensional pseudomanifold*.

Definition 2.4. Let L be a subcomplex of a finite simplicial complex K . A choice of one orientation for each simplex of K is a basis of $C_*^{\text{or}}(K)$ by Lemma 9.3 of *Homology*, and those belonging to the simplices of L span a subcomplex of $C_*^{\text{or}}(K)$ identified with $C_*^{\text{or}}(L)$. Say that a chain of $C_*^{\text{or}}(K)$ is *carried by L* if it lies in that subcomplex. Put

$$C_*^{\text{or}}(K, L) = C_*^{\text{or}}(K) / C_*^{\text{or}}(L),$$

again a chain complex of free abelian groups, and write $H_*^{\text{or}}(K, L)$ for its homology. A *relative cycle* is thus a chain $c \in C_*^{\text{or}}(K)$ with ∂c carried by L .

Example 2.5. In Example 1.6 put $\omega_i = (-1)^i[0, \dots, \widehat{i}, \dots, n+1]$. Then $\sum_i \omega_i$ is the expression for the boundary of the ordering $(0, 1, \dots, n+1)$, so the orientations cancel along every $(n-1)$ -simplex: the two terms of $\partial\partial[0, \dots, n+1]$ belonging to a given $(n-1)$ -face cancel. Thus $\partial\Delta^{n+1}$ is orientable.

Lemma 2.6. *Let K be an oriented n -dimensional pseudomanifold with boundary. Then*

$$z_K = \sum_s \omega_s \in C_n^{\text{or}}(K),$$

the sum over the n -simplices of K , satisfies

$$\partial z_K = \sum_t \partial_t \omega_{s(t)},$$

the sum over the free $(n-1)$ -simplices t of K , where $s(t)$ is the unique n -simplex of which t is a face. In particular ∂z_K is carried by ∂K , so that z_K is a relative cycle in $C_n^{\text{or}}(K, \partial K)$; and z_K is a cycle when K is closed.

Proof. By Definition 2.2,

$$\partial z_K = \sum_s \sum_{t \subset s} \partial_t \omega_s,$$

the inner sum over the codimension-one faces of s . Interchange the order of summation. Every codimension-one face of an n -simplex is an $(n-1)$ -simplex of K , and by Definition 1.1(ii) each such t lies in one or two n -simplices, so there are no terms beyond those listed. A non-free t contributes $\partial_t \omega_s + \partial_t \omega_{s'}$, which vanishes by Definition 2.3, and a free t contributes the single term $\partial_t \omega_{s(t)}$. Each free t is an $(n-1)$ -simplex of ∂K by Remark 1.4, so ∂z_K is carried by ∂K . $\square$

Proposition 2.7. *Let $n \geq 1$, let K be a normal oriented n -dimensional pseudomanifold with boundary, and suppose $\partial K \neq \emptyset$. For each $(n-1)$ -simplex t of ∂K put $\omega_t = \partial_t \omega_{s(t)}$. These orientations make ∂K an oriented closed $(n-1)$ -dimensional pseudomanifold, the induced orientation, and*

$$\partial z_K = z_{\partial K}.$$

Proof. By Theorem 1.14 the complex ∂K is a closed $(n-1)$ -dimensional pseudomanifold, and by Remark 1.4 its $(n-1)$ -simplices are exactly the free $(n-1)$ -simplices of K . Granting that the ω_t form an orientation, the displayed identity is then a restatement of Lemma 2.6.

For $n = 1$ there is nothing to verify, ∂K having no simplices of dimension $n-2$. Let $n \geq 2$ and let r be an $(n-2)$ -simplex of ∂K ; by Theorem 1.14 exactly two $(n-1)$ -simplices t and t' of ∂K have r as a face. Apply ∂ to Lemma 2.6:

$$0 = \partial \partial z_K = \sum_t \sum_{r' \subset t} \partial_{r'} \omega_t,$$

the outer sum over the $(n-1)$ -simplices of ∂K and the inner over their codimension-one faces, all of which are $(n-2)$ -simplices of ∂K . Terms belonging to distinct $(n-2)$ -simplices lie in distinct infinite cyclic summands of $C_{n-2}^{\text{or}}(K)$, so the terms belonging to r vanish on their own: $\partial_r \omega_t + \partial_r \omega_{t'} = 0$. This is Definition 2.3 for ∂K , which has no free $(n-2)$ -simplices. $\square$

Example 2.8. Take $K = \Delta^n$ as in Example 1.5, with the single orientation $\omega = [0, 1, \dots, n]$ of its one n -simplex; the condition of Definition 2.3 is empty, every $(n - 1)$ -simplex being free. Here $z_K = \omega$, and the induced orientation of $\partial\Delta^n$ is $\omega_i = (-1)^i[0, \dots, \widehat{i}, \dots, n]$, which is the orientation of Example 2.5 with n in place of $n + 1$. So $z_{\partial\Delta^n} = \partial[0, 1, \dots, n]$, as Proposition 2.7 requires.

Lemma 2.9. *Let $n \geq 1$ and let K be an n -dimensional pseudomanifold with boundary. Then $C_n^{\text{or}}(K, \partial K) = C_n^{\text{or}}(K)$ and there are no relative boundaries in degree n , so $H_n^{\text{or}}(K, \partial K)$ is the group of chains of $C_n^{\text{or}}(K)$ whose boundary is carried by ∂K , no two distinct such chains being homologous.*

Proof. K has no simplices of dimension $n + 1$, so $C_{n+1}^{\text{or}}(K) = 0$ and there are no relative boundaries; and ∂K has dimension at most $n - 1$, so $C_n^{\text{or}}(\partial K) = 0$ and $C_n^{\text{or}}(K, \partial K) = C_n^{\text{or}}(K)$. $\square$

Proposition 2.10. *Let $n \geq 1$ and let K be a strongly connected n -dimensional pseudomanifold with boundary. If K is orientable then $H_n^{\text{or}}(K, \partial K)$ is infinite cyclic, generated by the class of z_K ; if K is not orientable then $H_n^{\text{or}}(K, \partial K) = 0$. For closed K the same results hold with $H_n^{\text{or}}(K, \partial K)$ replaced by $H_n^{\text{or}}(K)$.*

Proof. By Lemma 2.9, $H_n^{\text{or}}(K, \partial K)$ is the group of chains $c \in C_n^{\text{or}}(K)$ whose boundary is carried by ∂K .

Choose an orientation η_s of each n -simplex; these form a basis of $C_n^{\text{or}}(K)$, so a chain is $c = \sum_s a_s \eta_s$ with $a_s \in \mathbb{Z}$, and as in the proof of Lemma 2.6 the component of ∂c at an $(n - 1)$ -simplex t is $a_s \partial_t \eta_s + a_{s'} \partial_t \eta_{s'}$ if t is not free, s and s' being the two n -simplices containing it, and $a_{s(t)} \partial_t \eta_{s(t)}$ if t is free. The components belonging to distinct t are independent, and by Remark 1.4 the $(n - 1)$ -simplices of ∂K are exactly the free ones. So ∂c is carried by ∂K if and only if $a_s \partial_t \eta_s = -a_{s'} \partial_t \eta_{s'}$ for every non-free t ; free $(n - 1)$ -simplices impose no condition. Since $\partial_t \eta_s = \pm \partial_t \eta_{s'}$ this says $a_{s'} = \pm a_s$, with the sign determined by t alone. Two distinct n -simplices with a common $(n - 1)$ -face share a non-free one, so the chains of Definition 1.8 cross only non-free $(n - 1)$ -simplices, and strong connectedness forces all the $|a_s|$ to be equal, say to $m \geq 0$.

If $m > 0$ then dividing by m and setting $\omega'_s = \text{sgn}(a_s) \eta_s$ produces an orientation $\{\omega'_s\}$ of K in the sense of Definition 2.3, and $c = m z'_K$ with $z'_K = \sum_s \omega'_s$. Now any two orientations of a strongly connected pseudomanifold with boundary differ by a global sign. Indeed, write $\omega'_s = \delta_s \omega_s$ with $\delta_s = \pm 1$; if s and s' share a codimension-one face t , necessarily non-free, then subtracting the cancellation conditions for the two orientations gives $(\delta_s - \delta_{s'}) \partial_t \omega_s = 0$, and $\partial_t \omega_s \neq 0$, so $\delta_s = \delta_{s'}$; strong connectedness then makes δ_s constant. Hence $z'_K = \pm z_K$ and $c = \pm m z_K$. So if K is not orientable the only relative cycle is 0, and if it is orientable the relative cycles are exactly the multiples of z_K , which is non-zero. $\square$

3. FUNDAMENTAL CLASS IN SINGULAR HOMOLOGY

We now pass from K to $|K|$, and from oriented simplicial homology to singular homology. The comparison is the chain map

$$\Theta_\lambda: C_*^{\text{or}}(K) \longrightarrow C_*(|K|)$$

of Definition 5.1 of *Further Properties*, attached to a total order λ on $V(K)$: it sends the orientation of a p -simplex s given by the increasing ordering $v_0 < \dots < v_p$ of its vertices to the affine singular simplex $\sigma_s = \langle v_0, \dots, v_p \rangle$. It is a chain map carrying the orientation of s into $C_*(|s|)$, by Lemma 5.2 there, and different orders give chain homotopic maps, by Lemma 5.3; by Theorem 5.5 it induces an isomorphism $H_*^{\text{or}}(K) \rightarrow H_*(|K|)$.

All of this passes to pairs. Let L be a subcomplex of K and give $V(L)$ the order restricted from λ . The orientation of a simplex s of L given by its increasing ordering is sent by Θ_λ to the affine simplex σ_s , whose image is $|s| \subseteq |L|$; so Θ_λ carries $C_*^{\text{or}}(L)$ into $C_*(|L|)$ and agrees there with the map built from L and the restricted order.

Lemma 3.1. *Let L be a subcomplex of a finite simplicial complex K and λ a total order on $V(K)$. Then Θ_λ descends to a chain map*

$$\Theta_\lambda: C_*^{\text{or}}(K, L) \longrightarrow C_*(|K|, |L|),$$

the induced map $\Theta_{\lambda}: H_*^{\text{or}}(K, L) \rightarrow H_*(|K|, |L|)$ is an isomorphism, and it is independent of λ .*

Proof. The chain map descends because Θ_λ carries $C_*^{\text{or}}(L)$ into $C_*(|L|)$, and it is then a map of short exact sequences of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*^{\text{or}}(L) & \longrightarrow & C_*^{\text{or}}(K) & \longrightarrow & C_*^{\text{or}}(K, L) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_*(|L|) & \longrightarrow & C_*(|K|) & \longrightarrow & C_*(|K|, |L|) & \longrightarrow & 0, \end{array}$$

the lower row being Proposition 6.2 of *Singular Homology*. By Theorem 4.3 and Proposition 4.4 of *Homology* this gives a ladder of long exact sequences with commuting squares, in which the maps attached to K and to L are isomorphisms by Theorem 5.5 of *Further Properties*. The five lemma, Exercise 4.6 of *Homology*, gives the assertion. For the independence, the chain homotopy D of Lemma 5.3 of those notes, between Θ_λ and $\Theta_{\lambda'}$ is constructed so that $D(\eta_s)$ is a chain in $|s|$; hence it carries $C_*^{\text{or}}(L)$ into $C_{*+1}(|L|)$ and descends to a chain homotopy on the quotients. $\square$

Theorem 3.2. *Let K be an oriented n -dimensional pseudomanifold with boundary and λ a total order on $V(K)$. For each n -simplex s let $\varepsilon_s = \pm 1$ according as ω_s is or is not the orientation given by the increasing ordering of the vertices of s . Then*

$$c_\lambda = \Theta_\lambda(z_K) = \sum_s \varepsilon_s \sigma_s \in C_n(|K|)$$

is a relative cycle for the pair $(|K|, |\partial K|)$, with $\partial c_\lambda = \Theta_\lambda(\partial z_K)$, and the class of c_λ in $H_n(|K|, |\partial K|)$ is non-zero. When K is normal, ∂z_K is the fundamental cycle $z_{\partial K}$ of the induced orientation of Proposition 2.7. For K closed the theorem says that c_λ is a cycle with non-zero class in $H_n(|K|)$.

Proof. Since Θ_λ is a chain map, $\partial c_\lambda = \Theta_\lambda(\partial z_K)$, and ∂z_K is carried by ∂K by Lemma 2.6; so ∂c_λ is a chain in $|\partial K|$, because Θ_λ carries the orientation of a simplex t into $C_*(|t|)$ by Lemma 5.2 of *Further Properties*. Thus c_λ is a relative cycle, and when $\partial K = \emptyset$ it is a cycle. The last assertion of the statement is Proposition 2.7.

For the non-vanishing, here are two arguments. The first is that K has no simplices of dimension $n+1$ and ∂K none of dimension n , so $C_{n+1}^{\text{or}}(K) = 0$ and $C_n^{\text{or}}(K, \partial K) = C_n^{\text{or}}(K)$; there are then no relative boundaries, and $H_n^{\text{or}}(K, \partial K)$ is the subgroup of the free abelian group $C_n^{\text{or}}(K)$ consisting of the chains whose boundary is carried by ∂K . The class of z_K there is the non-zero element z_K itself, and $\Theta_{\lambda*}$ is injective by Lemma 3.1.

The second uses only the class of the simplex, and locates the non-vanishing. Fix an n -simplex s and a point x of the open simplex of s . Then x lies in no simplex of dimension less than n , so $|\partial K| \subseteq |K| \setminus \{x\}$ and there is a map

$$H_n(|K|, |\partial K|) \longrightarrow H_n(|K|, |K| \setminus \{x\});$$

we compute the image of $[c_\lambda]$ under it. Every σ_t with $t \neq s$ has image $|t|$, which misses x , so all those terms die and c_λ goes to $\varepsilon_s \sigma_s$. Excising $|K| \setminus |s|$, whose closure misses x , identifies that group with $H_n(|s|, |s| \setminus \{x\})$, and the radial retraction of Lemma 1.3 of *Further Properties* replaces $|s| \setminus \{x\}$ by $|\partial s|$; since σ_s is a homeomorphism of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (|s|, |\partial s|)$, the group is infinite cyclic on $[\sigma_s]$ by Proposition 1.4 of those notes. So the image of $[c_\lambda]$ is a generator, and $[c_\lambda] \neq 0$. $\square$

Lemma 3.3. *Let K be an n -dimensional pseudomanifold with boundary, let λ be a total order on $V(K)$, and for each n -simplex s of K let x_s be a point of its open simplex. Then $H_n(|K|, |K| \setminus \{x_s\})$ is infinite cyclic, generated by the image of $[\sigma_s]$, and the image there of $\Theta_\lambda(\sum_r a_r \eta_r)$ is a_s times that generator, the η_r being the orientations given by the increasing orderings. Consequently the homomorphism*

$$H_n(|K|, |\partial K|) \longrightarrow \bigoplus_s H_n(|K|, |K| \setminus \{x_s\})$$

induced by the inclusions is injective, and its image consists of the families (a_s) for which the chain $\sum_s a_s \eta_s$ of $C_n^{\text{or}}(K)$ has boundary carried by ∂K ; equivalently, for which the singular chain $\sum_s a_s \sigma_s$ is a relative cycle of $(|K|, |\partial K|)$. So the local classes identify $H_n(|K|, |\partial K|)$ with the coherent families of local data, coherence being that condition, read on either side.

Proof. The point x_s lies in no simplex of dimension less than n , so $|\partial K| \subseteq |K| \setminus \{x_s\}$ and the map is defined. The computation is the one in the second proof of Theorem 3.2: excising $|K| \setminus |s|$, whose closure misses x_s , identifies $H_n(|K|, |K| \setminus \{x_s\})$ with $H_n(|s|, |s| \setminus \{x_s\})$; the radial retraction of Lemma 1.3 of *Further Properties* replaces $|s| \setminus \{x_s\}$ by $|\partial s|$; and σ_s is a homeomorphism of pairs $(\Delta^n, \partial \Delta^n) \rightarrow (|s|, |\partial s|)$, so the group is infinite cyclic on the image of $[\sigma_s]$ by Proposition 1.4 of those notes. A term $a_r \sigma_r$ with $r \neq s$ has image $|r|$, which misses x_s , and so dies; the term $a_s \sigma_s$ gives a_s times the generator.

For the last assertion, $\Theta_{\lambda*}$ carries $H_n^{\text{or}}(K, \partial K)$ isomorphically onto $H_n(|K|, |\partial K|)$ by Lemma 3.1, and by Lemma 2.9 that group is the group of chains $\sum_r a_r \eta_r$ of $C_n^{\text{or}}(K)$ whose boundary is carried by ∂K , no two of which are homologous. The display recovers every a_s from the local classes, so the map is injective and its image is exactly the set of families arising from such chains.

The two forms of the coherence condition agree. One way, Θ_λ is a chain map carrying $C_*^{\text{or}}(\partial K)$ into $C_*(|\partial K|)$ and $\Theta_\lambda(\sum_s a_s \eta_s) = \sum_s a_s \sigma_s$, so a boundary carried by ∂K goes to one carried by $|\partial K|$. The other way, write $\partial \sum_s a_s \eta_s = \sum_t b_t \xi_t$ over the $(n-1)$ -simplices t , with ξ_t the increasing orientations; then $\partial \sum_s a_s \sigma_s = \sum_t b_t \sigma_t$, the σ_t are distinct singular

simplices, that of t having image $|t|$, and $C_{n-1}(|\partial K|)$ is generated by the singular simplices with image in $|\partial K|$; so $b_t = 0$ unless t is a simplex of ∂K . $\square$

Remark 3.4. The injectivity has a geometric reading. If two classes have the same local class at every x_s , their difference dies in $H_n(|K|, |K| \setminus Z)$, where Z is the set of the x_s ; by the exact sequence of the triple $|\partial K| \subseteq |K| \setminus Z \subseteq |K|$ it is then carried by $|K| \setminus Z$. But $|K| \setminus Z$ deformation retracts onto $|K^{n-1}|$, each punctured open n -simplex retracting to its boundary, and a complex of dimension $n - 1$ carries no n -dimensional homology. So the difference vanishes: what is supported off the chosen points is supported on the $(n - 1)$ -skeleton, and there is nothing in degree n there.

Corollary 3.5. *If moreover K is normal, a class of $H_n(|K|, |\partial K|)$ is determined by its local class at a single point x_s , one n -simplex s being taken in each connected component of $|K|$.*

Proof. It is enough to treat one component, so let $|K|$ be connected; then K is strongly connected by Theorem 1.16. By Proposition 2.10 the group $H_n^{\text{or}}(K, \partial K)$ is infinite cyclic generated by z_K if K is orientable and is zero otherwise, so a class of $H_n(|K|, |\partial K|)$ is $m[K, \partial K]$ for a unique integer m , or is zero. In the notation of Lemma 3.3 this class is $\Theta_{\lambda*}$ of $m z_K = \sum_r m \varepsilon_r \eta_r$, so $a_s = m \varepsilon_s$ and m is determined by a_s , which the lemma recovers from the local class at x_s . $\square$

Definition 3.6. Let K be an oriented n -dimensional pseudomanifold with boundary. Its *fundamental class* is

$$[K, \partial K] = [c_\lambda] = \Theta_{\lambda*}[z_K] \in H_n(|K|, |\partial K|),$$

which by Lemma 3.1 does not depend on the total order λ used to define it. When K is closed, $|\partial K| = \emptyset$ and we write $[K] \in H_n(|K|)$.

Remark 3.7. Four comments. Reversing the orientation of K replaces z_K by $-z_K$ and hence $[K, \partial K]$ by $-[K, \partial K]$. Next, by Lemma 3.1 and Proposition 2.10, for a strongly connected orientable K the group $H_n(|K|, |\partial K|)$ is infinite cyclic generated by $[K, \partial K]$; for the closed pseudomanifold $K = \partial \Delta^{n+1}$ with the orientations of Example 2.5 this can be seen directly, c_λ being $\sum_i (-1)^i \delta^i = \partial \iota_{n+1}$, which generates $H_n(|K|) \cong \tilde{H}_n(S^n)$ by Proposition 1.4 of *Further Properties*. Third, if K is normal, so that ∂K is a closed pseudomanifold and carries a fundamental class, the connecting homomorphism of the pair takes the fundamental class of K to that of its boundary,

$$\partial [K, \partial K] = [\partial K] \in H_{n-1}(|\partial K|),$$

∂K carrying the induced orientation: the connecting homomorphism is computed by taking ∂ of a relative cycle, and $\partial c_\lambda = \Theta_\lambda(z_{\partial K})$ by Theorem 3.2, which is the cycle representing $[\partial K]$. Finally, orientability is genuinely needed only over $\mathbb{Z}$: for any n -dimensional pseudomanifold with boundary whatever, the chain $\sum_s \sigma_s$ with coefficients in $\mathbb{Z}/2$ is a relative cycle, because along each non-free $(n - 1)$ -simplex the two contributions agree rather than cancel and $1 + 1 = 0$, while the free ones lie in ∂K ; this gives a mod 2 fundamental class in $H_n(|K|, |\partial K|; \mathbb{Z}/2)$.

Exercise 3.8. Carry out Definition 3.6 for $K = \partial \Delta^2$ with the orientations of Example 2.5 and the order $0 < 1 < 2$, and identify c_λ explicitly as a sum of three singular 1-simplices in $|K| \cong S^1$.

Exercise 3.9. The *suspension* ΣK of a finite simplicial complex K has vertex set $V(K) \sqcup \{a, b\}$ and simplices those of K together with $\{a\}$, $\{b\}$ and $s \cup \{a\}$, $s \cup \{b\}$ for s a simplex of K . Show that if K is an n -dimensional pseudomanifold then ΣK is an $(n+1)$ -dimensional pseudomanifold, and that an orientation of K determines one of ΣK . Show also that ΣK is normal if and only if K is normal and, for $n \geq 1$, $|K|$ is connected. (Note that $\text{lk}_{\Sigma K}(s)$ is the suspension of $\text{lk}_K(s)$ for a simplex s of K , hence connected as soon as $\text{lk}_K(s) \neq \emptyset$; that $\text{lk}_{\Sigma K}(s \cup \{a\}) = \text{lk}_K(s)$; and that $\text{lk}_{\Sigma K}(\{a\}) = K$, which is where connectedness of $|K|$ enters.)

Exercise 3.10. Show that for $n \geq 1$ no n -dimensional pseudomanifold has exactly two n -simplices. (Count the incidences between n -simplices and $(n-1)$ -simplices; the two n -simplices would have the same codimension-one faces, hence the same vertices.) Two simplices glued along their whole boundary — the picture the count rules out — do not form a simplicial complex; compare Remark 4.10. Give an example of a 2-dimensional pseudomanifold that is not orientable.

4. GLUING SIMPLICIAL COMPLEXES ALONG FACES

All the simplicial topology of this section is in the service of one construction, that of §5: given a homology class of an arbitrary pair of spaces, it produces an oriented pseudomanifold with boundary carrying that class. The pseudomanifold is assembled by gluing copies of Δ^n to one another along their codimension-one faces, one copy for each singular simplex of a cycle representing the class, and which face is glued to which is dictated by the way the faces cancel in the boundary of that cycle. What has to be arranged is that the result be a simplicial complex, and the naive quotient is not one: two vertices of one simplex may be carried to the same class, and two distinct simplices may end up with the same set of vertices. Subdividing barycentrically twice repairs both, and once is not enough — Example 4.5 runs through the three stages for the simplest gluing of all, and Remark 4.10 says why no better choice of gluing evades the difficulty. Nothing about Δ^n enters until §4.3, so the first two subsections are written for a general complex.

4.1. Subdivision and orders of codimension-one faces. We first isolate the combinatorics of subdivision that is needed.

Lemma 4.1. *Let L be a finite simplicial complex in which every simplex is a face of an n -simplex. Then the same is true of the barycentric subdivision $\text{sd } L$, whose p -simplices are the flags $\tau_0 \subsetneq \tau_1 \subsetneq \cdots \subsetneq \tau_p$ of simplices of L . Moreover let $\Gamma: \tau_0 \subsetneq \cdots \subsetneq \tau_{n-1}$ be an $(n-1)$ -simplex of $\text{sd } L$. Then $\dim \tau_{n-1}$ is n or $n-1$, and the number of n -simplices of $\text{sd } L$ having Γ as a face is*

- (a) *exactly 2, if $\dim \tau_{n-1} = n$;*
- (b) *the number of n -simplices of L containing τ_{n-1} , if $\dim \tau_{n-1} = n-1$.*

Proof. Every flag of simplices of L extends to a flag whose members have all the dimensions $0, 1, \dots, n$: enlarge its top member to an n -simplex, possible by hypothesis, and insert faces of the intermediate dimensions. Hence every simplex of $\text{sd } L$ is a face of an n -simplex of $\text{sd } L$, which is the first assertion.

Let $\Gamma: \tau_0 \subsetneq \cdots \subsetneq \tau_{n-1}$ be an $(n-1)$ -simplex of $\text{sd } L$. The dimensions increase strictly along the flag, so $\dim \tau_{n-1} \geq n-1$, while no simplex of L has dimension greater than n ; so $\dim \tau_{n-1}$ is $n-1$ or n .

Suppose first that $\dim \tau_{n-1} = n-1$. Then the n strictly increasing dimensions $\dim \tau_0 < \cdots < \dim \tau_{n-1}$ must be $0, 1, \dots, n-1$, so nothing can be inserted below τ_0 or between two consecutive members of Γ , and the n -simplices of $\text{sd } L$ containing Γ are exactly the flags

$$\tau_0 \subsetneq \cdots \subsetneq \tau_{n-1} \subsetneq \sigma,$$

where σ is a simplex of L properly containing τ_{n-1} , hence an n -simplex of L containing τ_{n-1} . This establishes (b).

Otherwise $\dim \tau_{n-1} = n$, say $\tau_{n-1} = \sigma$, an n -simplex of L . The n dimensions $\dim \tau_i$ are then distinct elements of $\{0, 1, \dots, n\}$ with n among them, so there is a d with $0 \leq d \leq n-1$ for which

$$\dim \tau_i = i \quad (i < d), \quad \dim \tau_i = i + 1 \quad (i \geq d).$$

Consecutive members of Γ other than τ_{d-1} and τ_d differ by one in dimension, and nothing properly contains σ , so the only room for an insertion is between τ_{d-1} and τ_d , where, with the convention $\tau_{-1} = \emptyset$, the simplex τ_d has two more vertices than τ_{d-1} . An inserted simplex is therefore τ_{d-1} together with one of the two vertices of τ_d that are not vertices of τ_{d-1} , and both choices are faces of τ_d , hence simplices of L . There are exactly two ways to complete the flag, which establishes (a). $\square$

Corollary 4.2. *Let $n \geq 1$ and let $P = \text{sd}^2 \Delta^n$ be the second barycentric subdivision of the simplicial complex of all non-empty faces of Δ^n , so that $|P| = \Delta^n$. Then every simplex of P is a face of an n -simplex, and an $(n-1)$ -simplex of P is a face of exactly one n -simplex if it is contained in $\partial \Delta^n$, and of exactly two n -simplices otherwise. The subcomplex of P triangulating the face $\partial_i \Delta^n$ is sd^2 of that face.*

Proof. Apply Lemma 4.1 twice, first to Δ^n and then to $\text{sd } \Delta^n$. What is propagated at each stage is the statement that every simplex is a face of an n -simplex and that an $(n-1)$ -simplex is a face of exactly one n -simplex if it is contained in $\partial \Delta^n$ and of exactly two otherwise. This holds in Δ^n , viewed as a complex, whose only n -simplex is Δ^n itself and whose $(n-1)$ -simplices are the faces $\partial_i \Delta^n$. Suppose it holds in L , and let $\Gamma: \tau_0 \subsetneq \cdots \subsetneq \tau_{n-1}$ be an $(n-1)$ -simplex of $\text{sd } L$; then Γ is contained in $\partial \Delta^n$ precisely when τ_{n-1} is, the simplex Γ lying in τ_{n-1} and containing its barycenter. If $\dim \tau_{n-1} = n$ then $\tau_{n-1} \not\subset \partial \Delta^n$ and alternative (a) gives the count 2; if $\dim \tau_{n-1} = n-1$ then alternative (b) gives the number of n -simplices of L containing τ_{n-1} , which by the inductive hypothesis is 1 or 2 according as τ_{n-1} is or is not contained in $\partial \Delta^n$. The last statement of the corollary holds because barycentric subdivision takes subcomplexes to subcomplexes. $\square$

4.2. Triangulations of quotients of simplicial complexes. Here is the general form of the difficulty and of its repair.

Let M be a simplicial complex with vertex set V , the vertices of each simplex being ordered compatibly with taking faces, and let $\cong$ be an equivalence relation on the simplices of M generated by order-preserving isomorphisms between simplices of M , together with their restrictions to faces. For each j form the set $\bar{M}^j$ of equivalence classes of j -simplices. Because the generating isomorphisms preserve the orderings and are closed

under restriction to faces, there are well-defined face maps

$$\partial_{i_1} \circ \cdots \circ \partial_{i_r} : \bar{M}^j \longrightarrow \bar{M}^{j-r}$$

satisfying the usual commutation relations $\partial_i \partial_j = \partial_{j-1} \partial_i$ for $i < j$. In particular there are $j + 1$ maps $\bar{M}^j \rightarrow \bar{M}^0 = \bar{V}$, and there is no reason for any of these face maps to be injective. The collection $\{\bar{M}^j\}$ together with its face maps is a combinatorial object $\bar{M}$ generalizing a simplicial complex; the superscript in $\bar{M}^j$ marks the dimension of the simplices in it, and is not a skeleton. Such an object *is a simplicial complex* if there is a bijection of the union of the $\bar{M}^j$ with the set of simplices of a simplicial complex, sending k -simplices to k -simplices and commuting with the face relations. We write $\Delta(\bar{V})$ for the full simplex on $\bar{V}$, the simplicial complex of all non-empty finite subsets of $\bar{V}$.

Lemma 4.3. *With M and $\cong$ as above, the quotient object $\bar{M}$ is a simplicial complex if and only if it meets the following two conditions.*

- (i) *The image in $\bar{V}$ of any $[\sigma] \in \bar{M}^k$ has cardinality $k + 1$; that is, the map from the vertices of $[\sigma]$ to $\bar{V}$ is injective.*
- (ii) *Granting (i), each $[\sigma] \in \bar{M}^k$ has a k -simplex as image in $\Delta(\bar{V})$, namely the one spanned by the images of its vertices; the condition is that distinct simplices of $\bar{M}$ have distinct images, that is, that a simplex of $\bar{M}$ is determined by its set of vertices in $\bar{V}$.*

Proof. Condition (i) gives a map $\bar{M} \rightarrow \Delta(\bar{V})$, compatible with the face maps, that is an isomorphism on each simplex. Its image is automatically a subcomplex of $\Delta(\bar{V})$, the faces of the image of $[\sigma]$ being the images of the faces of $[\sigma]$. What (ii) adds is that $\bar{M}$ maps bijectively onto that image. Granting both, we have a combinatorial isomorphism from $\bar{M}$ to a simplicial complex, carrying k -simplices to k -simplices and commuting with the face relations. The converse is obvious. $\square$

Now subdivide. Let M_1 be the barycentric subdivision of M and let M_2 be the barycentric subdivision of M_1 . There are two rank functions on vertices. A vertex of M_1 is the barycenter of a simplex of M , and rk_1 of that vertex is the dimension of that simplex; a vertex of M_2 is the barycenter of a simplex of M_1 , and rk_2 of that vertex is the dimension of that simplex. Each simplex of M_1 has its vertices totally ordered by rk_1 , and each simplex of M_2 has its vertices totally ordered by rk_2 . The relation $\cong$ induces relations on M_1 and on M_2 preserving rk_1 and rk_2 respectively, the second because an isomorphism between simplices of M_1 preserves their dimension; so these relations too are generated by vertex-order-preserving isomorphisms, and we may form $\bar{M}_1$ and $\bar{M}_2$, which carry the combinatorial structure described above.

For a simplex σ of M_1 let $\text{RK}_1(\sigma)$ be the set of rk_1 of the vertices of σ , and give the vertex of M_2 that is the barycenter of σ the same set RK_1 ; thus rk_2 of that vertex is the cardinality of RK_1 minus one. The generating isomorphisms preserve rk_1 , hence preserve RK_1 , so every vertex of $\bar{M}_2$ has a well-defined RK_1 .

Theorem 4.4. *Let M and $\cong$ be as at the beginning of this subsection, let M_1 and M_2 be the first and second barycentric subdivisions of M , and let $\bar{M}_1$ and $\bar{M}_2$ be the quotient objects they determine. Then the vertices of every simplex of $\bar{M}_1$ stay distinct, which is condition (i) of Lemma 4.3; and $\bar{M}_2$ satisfies both conditions of that lemma and is therefore a simplicial complex.*

Proof. Step 1: for every simplex σ of M_1 the vertices of σ remain distinct in $\bar{M}_1$. The equivalence relation is generated by simplicial isomorphisms of M and hence preserves rk_1 . Since distinct vertices of a simplex of M_1 have distinct rk_1 , they are never identified with one another. The same argument with M_1 in place of M , and rk_2 in place of rk_1 , shows that distinct vertices of a simplex of M_2 are never identified in $\bar{M}_2$; that is, $\bar{M}_2$ also satisfies condition (i).

Still, $\bar{M}_1$ need not be a simplicial complex, because condition (ii) may fail.

Step 2: $\bar{M}_2$ satisfies condition (ii). We use the following observation. Distinct vertices of a simplex σ of M_1 have distinct rk_1 , and a face of σ is spanned by a subset of the vertices of σ ; hence a face of σ is determined by the set of rk_1 of its vertices, that is, by its RK_1 .

Let τ and τ' be simplices of $\bar{M}_2$ with the same set of vertices, and represent them by simplices of M_2 . A p -simplex of M_2 is a flag of simplices of M_1 , so τ is represented by $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ and τ' by $\sigma'_0 < \sigma'_1 < \cdots < \sigma'_p$, the vertices being the classes of the barycenters of the σ_r and of the σ'_r . The cardinality of $\text{RK}_1(\sigma_r)$ is $\dim \sigma_r + 1$, which increases with r , so the sets $\text{RK}_1(\sigma_0), \dots, \text{RK}_1(\sigma_p)$ are distinct, and likewise for τ' . Since RK_1 is defined on the vertices of $\bar{M}_2$, the identification of the vertices of τ with those of τ' therefore matches the barycenter of σ_r with that of σ'_r , and

$$\text{RK}_1(\sigma_r) = \text{RK}_1(\sigma'_r) \quad \text{for every } r.$$

In particular the barycenters of σ_p and of σ'_p have the same class in $\bar{M}_2$, so there is a chain of relations given by linear isomorphisms carrying σ_p to σ'_p ; let $g: \sigma_p \rightarrow \sigma'_p$ be the composite of that chain. It carries faces of σ_p to faces of σ'_p and preserves rk_1 , hence preserves RK_1 . Fix $r < p$. Then $g\sigma_r$ and σ'_r are faces of σ'_p with

$$\text{RK}_1(g\sigma_r) = \text{RK}_1(\sigma_r) = \text{RK}_1(\sigma'_r),$$

so $g\sigma_r = \sigma'_r$ by the observation. Hence g carries the flag representing τ to the flag representing τ' , and $\tau = \tau'$ in $\bar{M}_2$. Lemma 4.3 now applies. $\square$

Example 4.5. Both subdivisions are needed already for $M = \Delta^1$, the complex of the two vertices e_0, e_1 and the edge joining them, with the relation generated by the order-preserving isomorphism $e_0 \mapsto e_1$ of 0-simplices.

In $\bar{M}$ the edge has both of its vertices in the single class $v = [e_0] = [e_1]$, so condition (i) of Lemma 4.3 fails.

In $M_1 = \text{sd } M$ the edge is cut at its barycenter b into two edges $\{e_0, b\}$ and $\{b, e_1\}$. Their vertices stay distinct in $\bar{M}_1$, which is condition (i), as Theorem 4.4 says; but both edges acquire the vertex set $\{v, b\}$, so condition (ii) fails and $\bar{M}_1$ is not a simplicial complex. It is the union of two 1-simplices with their boundaries glued together.

In $M_2 = \text{sd}^2 M$ the interval is cut into four edges, with vertices e_0, c_1, b, c_2, e_1 in order. The quotient has the four vertices v, c_1, b, c_2 and the four edges joining consecutive ones, distinct vertex sets all: $\bar{M}_2$ is the circle triangulated with four edges.

4.3. Gluing copies of a simplex. Now specialize: the pieces are copies of one simplex, subdivided twice, and the identifications pair off their codimension-one faces.

Lemma 4.6. *Let $P = \text{sd}^2 \Delta^n$ be as in Corollary 4.2, let $P_1, \dots, P_N$ be copies of P , and let a fixed-point-free involution be given on a subset of the set of pairs (k, i) with $1 \leq k \leq N$*

and $0 \leq i \leq n$, the pairs outside that subset being left unglued. Glue, for each pair $\{(k, i), (l, j)\}$ of the involution, the subcomplex of P_k triangulating $\partial_i \Delta^n$ to the subcomplex of P_l triangulating $\partial_j \Delta^n$, by the simplicial isomorphism induced by the affine identification $\delta^j \circ (\delta^i)^{-1}$. Then the result is a finite simplicial complex K , and the quotient map

$$q: \coprod_{k=1}^N \Delta_k^n \longrightarrow |K|$$

is the quotient by the corresponding identifications of the boundary faces. When the involution is defined on all of the pairs, every boundary face of every copy is glued to another.

Proof. Apply Theorem 4.4 with $\coprod_{k=1}^N \Delta_k^n$ in place of M , taken with the standard ordering of the vertices of each copy, and with the equivalence relation generated by the identifications of the codimension-one faces prescribed by the involution, together with their restrictions to faces; these identifications preserve the vertex orderings, being induced by the maps $\delta^j \circ (\delta^i)^{-1}$. The second barycentric subdivision of $\coprod_k \Delta_k^n$ is $\coprod_k P_k$, so the theorem says that the quotient is a simplicial complex; this is the complex K of the statement. Finally $|P_k| = \Delta_k^n$, and for finite complexes the realization of the quotient complex is the quotient of the realizations, so q is the quotient map by the prescribed identifications of the boundary faces. $\square$

Lemma 4.7. *Let L be a finite simplicial complex realized rectilinearly in a Euclidean space: its vertices are points, each simplex is the convex hull of its vertices, and two simplices meet in a common face or not at all. Suppose that the union C of its simplices is convex and that its affine span has dimension n , dimensions and interiors below being taken in that span, and let t be a simplex of L of dimension $d \leq n - 2$. Then $\text{lk}(t)$ is connected.*

Proof. Let $\text{st}(t)$ be the subcomplex of L consisting of the simplices containing t and all of their faces. Every point of $|\text{st}(t)|$ lies in $|s|$ for some simplex $s \supseteq t$, and for such an s the vertex set splits as that of t together with that of a simplex $u = s \setminus t$ of $\text{lk}(t)$, so that a point of $|s|$ is written $y = \sum_{v \in t} \alpha_v v + \sum_{w \in u} \beta_w w$. Put $\lambda(y) = \sum_{w \in u} \beta_w$; then $y \in |t|$ exactly when $\lambda(y) = 0$, and for $y \notin |t|$ the point

$$\pi(y) = \lambda(y)^{-1} \sum_{w \in u} \beta_w w \in |\text{lk}(t)|$$

is independent of the simplex s chosen and depends continuously on y . This defines a continuous map $\pi: |\text{st}(t)| \setminus |t| \rightarrow |\text{lk}(t)|$.

Let x be a point of the interior of $|t|$ and let A be the d -dimensional affine subspace spanned by t . The union of the closed simplices of L not lying in $\text{st}(t)$ is a closed set missing x , so $|\text{st}(t)|$ is a neighbourhood of x in C ; and $|t|$ is a neighbourhood of x in A . Choose $\varepsilon > 0$ so small that the convex set $B = \{y \in C : |y - x| \leq \varepsilon\}$ lies in $|\text{st}(t)|$ and meets A inside $|t|$.

We claim $B \setminus A$ is connected. Let $p, q \in B \setminus A$. The points r for which the segment $[p, r]$ meets A lie in the affine span of $\{p\} \cup A$, of dimension at most $d + 1 \leq n - 1$, and similarly for q . Since B has dimension n its interior is a non-empty open set, which cannot be covered by two proper affine subspaces; so there is an $r \in B$ with $[p, r]$ and $[r, q]$ both disjoint from A . These segments lie in B by convexity, and the claim follows.

Finally $\pi(B \setminus A) = |\text{lk}(t)|$. Indeed, given $b \in |u|$ with $u \in \text{lk}(t)$, the points $y_\mu = (1 - \mu)x + \mu b$ lie in $|t \cup u|$ and satisfy $\pi(y_\mu) = b$ for $\mu > 0$; for small $\mu > 0$ they lie in B , and they are not in A because $B \cap A \subseteq |t|$. Thus $|\text{lk}(t)|$ is the continuous image of a connected set. $\square$

Proposition 4.8. *Let $n \geq 1$ and let K be a complex obtained, as in Lemma 4.6, by gluing copies $P_1, \dots, P_N$ of $P = \text{sd}^2 \Delta^n$ to one another along a pairing of some of their codimension-one faces. Then K is an n -dimensional pseudomanifold with boundary, its free $(n - 1)$ -simplices being exactly those lying on the unpaired facets; and it is normal in the sense of Definition 1.10, the link of every simplex of K of dimension at most $n - 2$ being connected.*

Remark 4.9. This is what makes the complexes constructed in the proof of Theorem 5.1 and in *The Hurewicz theorems* pseudomanifolds with boundary, and normal ones, so that Theorem 1.14 and Proposition 2.7 apply to them.

Proof. Condition (i) of Definition 1.1 holds because every simplex of K lies in a copy of P and is a face of an n -simplex there, by Corollary 4.2. For (ii), let t be an $(n - 1)$ -simplex of K , and let $\tilde{t}$ be an $(n - 1)$ -simplex of some copy P_k mapping to it. Since P_k is a copy of Δ^n , the simplex $\tilde{t}$ is contained in at most one of the codimension-one faces $\partial_i \Delta^n$. By Corollary 4.2, if $\tilde{t}$ is interior to P_k then it is a face of two n -simplices of P_k ; otherwise it is contained in a facet of P_k and is a face of only one n -simplex of P_k . If it is interior, it is identified with nothing and t is a face of two n -simplices. If it lies in a paired facet, the simplex matched with it in the partner also maps to t , and the two contribute one n -simplex each. If it lies in an unpaired facet, it is identified with nothing and t is a face of one n -simplex. So every $(n - 1)$ -simplex of K is a face of one or two, and the free ones are exactly those lying on the unpaired facets.

There remains the normality. Write $\rho: \coprod_k P_k \rightarrow K$ for the simplicial quotient map. By Lemma 4.3(i) the vertices of a simplex stay distinct in the quotient, so ρ carries each simplex isomorphically onto a simplex of the same dimension.

Let t be a simplex of K with $\dim t = d \leq n - 2$ and let $t_1, \dots, t_r$ be the simplices of $\coprod_k P_k$ lying over t ; they form a single equivalence class. Set $L_i = \rho(\text{lk}(t_i))$. If $u \in \text{lk}(t_i)$ then $\rho(t_i \cup u) = t \cup \rho(u)$ is a simplex of dimension $d + \dim u + 1$, so $\rho(u)$ is disjoint from t and L_i is a subcomplex of $\text{lk}(t)$.

First, $\text{lk}(t) = \bigcup_i L_i$. A simplex u of $\text{lk}(t)$ gives the simplex $t \cup u$ of K ; lift it to a simplex s of some P_k . As ρ is injective on s , the vertex set of s splits as $t_i \cup u'$ with $\rho(t_i) = t$ and $\rho(u') = u$, and then $u \in L_i$.

Second, each L_i is connected: it is the image of $\text{lk}(t_i)$, and that is connected by Lemma 4.7, whose hypotheses hold for $P_k = \text{sd}^2 \Delta^n$: it is rectilinear, being a subdivision of Δ^n , and the union of its simplices is Δ^n , convex with affine span of dimension n .

Third, the union is connected. The equivalence relation is generated by the prescribed isomorphisms between the subcomplexes triangulating paired facets, together with their restrictions to faces, so any two of $t_1, \dots, t_r$ are joined by a chain of single such identifications; it is enough to see that $L_i \cap L_j \neq \emptyset$ whenever t_i and t_j are directly identified by one of them. Say φ identifies the subcomplex Q of P_k triangulating one facet with the subcomplex Q' of P_l triangulating its partner, and $\varphi(t_i) = t_j$. Now Q is a subdivision of

an $(n-1)$ -simplex and so is pure of dimension $n-1$; choose an $(n-1)$ -simplex F of Q containing t_i and put $F' = \varphi(F) \supseteq t_j$. Then $F \setminus t_i$ is a simplex of $\text{lk}(t_i)$ of dimension $n-d-2 \geq 0$, hence non-empty, and φ carries it to $F' \setminus t_j$. The two therefore have the same image under ρ , which is a simplex of $L_i \cap L_j$. $\square$

Remark 4.10. Some subdivision is unavoidable, and one is not enough. The proof of Theorem 4.4 shows where a single subdivision fails: two distinct faces of a simplex can be identified with one another, whereas after a second subdivision a face is pinned down by the ranks of its vertices, which the identifications preserve. Example 4.5 is that failure in its smallest instance. Self-gluing of the kind occurring there cannot be avoided by a better choice of the pairing in Step 2 of the proof of Theorem 5.1: for n odd the map $\sigma: \Delta^n \rightarrow \Delta^n / \partial \Delta^n$ is by itself a cycle, all of whose faces are the same constant singular simplex, so every pairing of its facets glues σ to itself.

5. ORIENTED PSEUDOMANIFOLDS REPRESENT HOMOLOGY

The construction of §§2–3 turns a piece of geometry into a relative homology class. The theorem of this section is the converse, and is the reason for introducing pseudomanifolds at all: nothing is lost, since every integral homology class of every pair of spaces is the image of the fundamental class of an oriented pseudomanifold with boundary. The absolute statement is the case $Y = \emptyset$, recorded as Corollary 5.2. The proof has one geometric input, the gluing of §4, and one homological one, the local description of relative n -classes in Lemma 3.3.

Theorem 5.1. *Let (X, Y) be a pair of spaces, let $n \geq 0$ and let $\alpha \in H_n(X, Y)$. Then there are an oriented n -dimensional pseudomanifold with boundary K and a map of pairs*

$$f: (|K|, |\partial K|) \longrightarrow (X, Y)$$

with

$$f_*[K, \partial K] = \alpha.$$

Corollary 5.2. *Let X be a space, let $n \geq 0$ and let $\alpha \in H_n(X)$. Then there are an oriented closed n -dimensional pseudomanifold K and a map $f: |K| \rightarrow X$ with*

$$f_*[K] = \alpha.$$

Proof. Apply Theorem 5.1 to the pair $(X, \emptyset)$, for which $H_n(X, \emptyset) = H_n(X)$. A map of pairs carries $|\partial K|$ into $\emptyset$, so $|\partial K| = \emptyset$ and K is closed; and then $[K, \partial K] = [K]$ in $H_n(|K|, \emptyset) = H_n(|K|)$. $\square$

Proof of Theorem 5.1. For $n = 0$ every 0-chain is a relative cycle, so a class of $H_0(X, Y)$ is represented by a chain $\sum_{k=1}^N \rho_k x_k \in C_0(X)$ with $\rho_k = \pm 1$ and $x_k \in X$, and one takes for K the 0-dimensional pseudomanifold with N points, oriented by the ρ_k , mapped to X by x_k ; its boundary is empty, so f is a map of pairs. So let $n \geq 1$.

Step 1: a representative with coefficients ± 1 . Write $\alpha = [z]$ with $z \in C_n(X)$ a relative cycle, that is, a chain with $\partial z \in C_{n-1}(Y)$. Repeating each singular simplex as often as its coefficient requires, and using $\sigma - \sigma = 0$ to avoid the empty sum when $\alpha = 0$, we may write

$$z = \sum_{k=1}^N \rho_k \sigma_k, \quad \rho_k = \pm 1, \quad N \geq 1,$$

where the singular n -simplices $\sigma_k: \Delta^n \rightarrow X$ need not be distinct.

Step 2: pairing the faces. Call the pairs (k, i) with $1 \leq k \leq N$ and $0 \leq i \leq n$ the *facets*, and attach to the facet (k, i) the singular $(n-1)$ -simplex $\tau_{k,i} = \sigma_k \circ \delta^i$ and the sign $\eta_{k,i} = \rho_k(-1)^i$. Then

$$\partial z = \sum_{k,i} \eta_{k,i} \tau_{k,i} \in C_{n-1}(Y).$$

The group $C_{n-1}(X)$ is free on the singular $(n-1)$ -simplices of X , and $C_{n-1}(Y)$ is the subgroup generated by those whose image lies in Y . Fix a singular $(n-1)$ -simplex τ , let F_τ be the set of facets with $\tau_{k,i} = \tau$, and let m_τ be the sum of the signs $\eta_{k,i}$ over F_τ ; this is the coefficient of τ in ∂z , so

$$m_\tau \neq 0 \implies \tau(\Delta^{n-1}) \subseteq Y.$$

Within F_τ pair off facets of opposite sign for as long as both signs occur. Since the signs are ± 1 , this leaves exactly $|m_\tau|$ facets of F_τ unpaired, all carrying the same sign, and leaves none when $m_\tau = 0$. Doing this for every τ produces a fixed-point-free involution θ , defined on the set of *paired* facets, with

$$\tau_{\theta(k,i)} = \tau_{k,i}, \quad \eta_{\theta(k,i)} = -\eta_{k,i},$$

and leaves a set of *free* facets, each of which carries a singular simplex with image in Y . The paired facets contribute nothing to ∂z , so

$$\partial z = \sum_{(k,i) \text{ free}} \eta_{k,i} \tau_{k,i}.$$

When $Y = \emptyset$ every m_τ is zero and there are no free facets.

Step 3: the complex and the map. Let K be the simplicial complex obtained from N copies of $P = \text{sd}^2 \Delta^n$ by gluing the facets in the pairs of θ and leaving the free facets unglued, as in Lemma 4.6, and let $q_k: \Delta^n \rightarrow |K|$ be the restriction of the quotient map to the k -th copy. Since $\tau_{k,i} = \tau_{l,j}$ for paired facets, the maps σ_k agree on identified points, so there is a unique map

$$f: |K| \longrightarrow X, \quad f \circ q_k = \sigma_k \quad (1 \leq k \leq N).$$

By Proposition 4.8, K is a normal n -dimensional pseudomanifold with boundary, whose free $(n-1)$ -simplices are exactly those lying on the free facets.

Consequently $|\partial K|$ is the union of the images under q of the free facets: the subcomplex of P_k triangulating a free facet is pure of dimension $n-1$, so each of its simplices is a face of a free $(n-1)$ -simplex of K , and conversely every simplex of ∂K lies on a free facet. On such a facet f takes the value $f(q_k(\delta^i y)) = \sigma_k(\delta^i y) = \tau_{k,i}(y)$, which lies in Y by Step 2. Hence f is a map of pairs $(|K|, |\partial K|) \rightarrow (X, Y)$.

Step 4: orientations. Regard each q_k as a singular n -simplex of $|K|$ and put

$$u = \sum_{k=1}^N \rho_k q_k \in C_n(|K|).$$

Then

$$\partial u = \sum_{k,i} \eta_{k,i} (q_k \circ \delta^i) = \sum_{(k,i) \text{ free}} \eta_{k,i} (q_k \circ \delta^i),$$

because paired facets are glued precisely so that $q_k \circ \delta^i = q_l \circ \delta^j$ when $\theta(k, i) = (l, j)$, while $\eta_{k,i} = -\eta_{l,j}$; and every surviving term has image in $|\partial K|$ by Step 3. So u is a relative cycle for the pair $(|K|, |\partial K|)$.

The idea of what follows is that each copy Δ_k^n carries the orientation of Δ^n ; this induces an orientation on every n -simplex of K lying in it, and those, corrected by the signs ρ_k , are the ω_s produced below. Nothing has to be compared, however: all that is checked is that the local class of u at an interior point of an n -simplex is a *generator*, and which generator it is then defines ω_s . Only u , its being a relative cycle, and its local classes are singular; orientations are simplicial data, and the comparison of §3 is what relates the two.

Fix an n -simplex s of K and a point x_s of its open simplex. The gluings identify only codimension-one faces and their faces, so no two n -simplices of the copies are identified and s comes from an n -simplex of exactly one copy, say the k -th; and the open simplex of s lies in the interior of Δ_k^n . So x_s has a single preimage y under q_k and none under q_l for $l \neq k$, and y lies in the interior of Δ_k^n . An open subset of the interior of a copy is saturated for the quotient map, so q_k carries a small open neighbourhood V of y homeomorphically onto an open neighbourhood W of x_s in $|K|$; excising the complements of V and of W identifies both $H_n(\Delta^n, \Delta^n \setminus \{y\})$ and $H_n(|K|, |K| \setminus \{x_s\})$ with $H_n(W, W \setminus \{x_s\})$, so q_{k*} is an isomorphism between them. The image of $[\iota_n]$ in $H_n(\Delta^n, \Delta^n \setminus \{y\})$ is a generator, by Lemma 1.3 and Proposition 1.4 of *Further Properties*. The terms $\rho_l q_l$ of u with $l \neq k$ are chains in $|K| \setminus \{x_s\}$ and so die there; hence the local class of u at x_s is ρ_k times a generator of $H_n(|K|, |K| \setminus \{x_s\})$.

Fix a total order λ on $V(K)$, let η_s be the orientation of s given by the increasing ordering of its vertices, and let σ_s be the corresponding affine singular simplex. By Lemma 3.3 the local class of σ_s at x_s generates $H_n(|K|, |K| \setminus \{x_s\})$, so the local class of u there, being a generator too, is a_s times it with $a_s = \pm 1$. Set $\omega_s = a_s \eta_s$: the orientation of s whose affine simplex has at x_s the local class that u has there. This is done for every n -simplex s of K .

That the ω_s orient K now needs no computation. The family (a_s) is the family of local data of a class, namely of $[u]$, so it is coherent in the sense of Lemma 3.3: the chain $\sum_s a_s \eta_s = \sum_s \omega_s$ has boundary carried by ∂K . As in the proof of Proposition 2.10 that says exactly that $\partial_t \omega_s + \partial_t \omega_{s'} = 0$ at every non-free $(n-1)$ -simplex t , with s and s' the two n -simplices containing it, which is Definition 2.3. So K is oriented, $\sum_s \omega_s$ is the relative oriented simplicial cycle z_K of $(K, \partial K)$, and

$$\Theta_{\lambda*}[z_K] = [u].$$

Step 5: the class. By Definition 3.6 the left-hand side above is $[K, \partial K]$, so u represents the fundamental class. Since $f \circ q_k = \sigma_k$,

$$f_{\#}(u) = \sum_k \rho_k (f \circ q_k) = \sum_k \rho_k \sigma_k = z,$$

and f is a map of pairs, so

$$f_*[K, \partial K] = [f_{\#}u] = [z] = \alpha. \quad \square$$

Remark 5.3. Let K and f be as in Theorem 5.1 and let $g: |\partial K| \rightarrow Y$ be the restriction of f . That K is normal, by Proposition 4.8, makes ∂K a closed pseudomanifold with an

induced orientation, and then, with ∂K so oriented,

$$g_*[\partial K] = \partial \alpha \in H_{n-1}(Y),$$

where ∂ is the connecting homomorphism of the pair (X, Y) . Indeed the connecting homomorphism is natural for maps of pairs, by Proposition 6.5 of *Singular Homology*, and $\partial[K, \partial K] = [\partial K]$ by Remark 3.7. So the construction represents a relative class and its boundary at the same time.

Remark 5.4. The theorem is what makes pseudomanifolds the simplest geometric representatives of homology classes, and its proof is elementary: the pseudomanifold is nothing but the cycle, with its simplices glued along the faces that cancel. The corresponding question for closed oriented manifolds — Steenrod’s realizability problem — is much harder and has a negative answer; Thom showed that every class of $H_*(X; \mathbb{Z}/2)$ is carried by a closed manifold, that integral classes of degree at most 6 are carried by closed oriented manifolds, and that in degree 7 there are integral classes that are not, although some non-zero multiple of any class always is. The singularities that appear in Theorem 5.1 are therefore essential.

Exercise 5.5. Let $X = S^1 \vee S^1$ and let $\alpha \in H_1(X)$ be the sum of the two generators. Carry out Steps 1–3 for the evident representing cycle, and identify K and f .

Exercise 5.6. Let X be path connected and let $n \geq 1$. Show that in Theorem 5.1 one may arrange that $|K|$ be connected, at the cost of normality. (First move the images of finitely many vertices to a common point along paths, deforming f ; then glue the components of K at those vertices, as in Remark 1.11, the gluing being carried out on the quotient as in Lemma 4.6. Check that the fundamental class of the result maps to α .) The complex so obtained is connected but in general not strongly connected, so by Theorem 1.16 it is not normal; the price of connecting the components at vertices is that Theorem 1.14 and Proposition 2.7 no longer apply to it.

Exercise 5.7. State and prove the mod 2 analogue of Theorem 5.1, using the mod 2 fundamental class of Remark 3.7; observe that no orientation and hence no sign bookkeeping is needed in Step 2.