

PROBLEM SET 3

Problem 1. Show that

$$H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = \begin{cases} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

Show that if $j: \Delta^n \rightarrow \mathbb{R}^n$ is a map with $j^{-1}(0) = x \in \text{int}(\Delta^n)$ and if j is smooth in a neighborhood of x and Dj_x is a linear isomorphism then the singular simplex j is a relative cycle and generates $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. Given such a $j: \Delta^n \rightarrow \mathbb{R}^n$ show that for all $y \in \mathbb{R}^n$ sufficiently close to 0, the map j is also a relative cycle for $(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ and a generator the the relative n^{th} -homology group .

Problem 2. Show that if $k: \Delta^n \rightarrow \mathbb{R}^n$ has the same properties as j in Problem 1 then the relative homology classes $[j]$ and $[k]$ represented by j and k differ by a multiplicative factor which is $\text{sign}(\det(Dk_x \circ (Dj_x)^{-1}))$. Show that given j and k with these properties, and for all y sufficiently close to 0, the multiplicative factor comparing the class of $[j]$ and $[k]$ in $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ is constant. We call local orientations in some small neighborhood of 0 *consistent*.

Problem 3. Let M be a smooth n -manifold with an orientation. This means that there is a distinguished maximal sub-atlas of M of coordinate charts $\varphi_U: U \rightarrow V \subset \mathbb{R}^n$ between open sets U in M and V in $\mathbb{R}^n$ such that for each pair $\varphi_{U_1}: U_1 \rightarrow V_1$ and $\varphi_{U_2}: U_2 \rightarrow V_2$ the map

$$\varphi_{U_2}(U_1 \cap U_2) \xrightarrow{\varphi_{U_2}^{-1} \circ \varphi_{U_1}} \varphi_{U_1}(U_1 \cap U_2)$$

is orientation-preserving; i.e., has positive determinant at every point where it is defined. Generalize the definition in Problem 2 to define the homology orientation of M at a point $p \in M$ and show that these orientations are locally consistent. Conversely, show that if M has a family of homology orientations at each of its points and these are locally consistent, then they determine an orientation of M in the differential-topological sense.

Problem 4. Suppose that $\sigma: \Delta^n \rightarrow \Delta^n$ is a continuous map with the property that $\sigma(\partial\Delta^n) \subset \Delta^n$ and that on the pre-image $\sigma^{-1}(U)$ of some open neighborhood U of the barycenter b of Δ^n , the map σ is smooth and for every $y \in \sigma^{-1}(b)$, the differential $(D\sigma)_y$ is a linear isomorphism. Compute the relative homology class represented of $(\Delta^n, \partial\Delta^n)$ represented by σ .

Problem 5. Show for a finite simplicial complex of dimension n that the homology vanishes above dimension n . Show that the rank of the homology of dimension k is bounded above by the number of k -simplices in the complex.

Problem 6. Compute the integral homology of $\mathbb{R}P^n$. Compute its homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Problem 7. I used the idea of small chains to prove both the Mayer-Vietoris Theorem and Excision. Formally, deduce each from the other using only the homology statements; i.e., without resorting to chain arguments.