

PROBLEM SET 2

Problem 1. Define the complex projective plane $\mathbb{CP}^2$ as the space of complex one-dimensional subspaces of $\mathbb{C}^3$ with the natural topology: Given a complex line L and an open subset $U \subset \mathbb{C}^3$ meeting L , the set of complex lines meeting U is an open neighborhood of $[L]$, and all open neighborhoods of $[L]$ in $\mathbb{CP}^2$ are of this form. Show that the lines contained in the subspace $\mathbb{C}^2 = \{z_3 = 0\} \subset \mathbb{C}^3$ form a closed subset homeomorphic to S^2 , denoted $\mathbb{CP}^1$ and that its complement is homeomorphic to $\mathbb{C}^2$. Construct a map from the closed unit disk D^4 to $\mathbb{CP}^2$ whose interior maps homeomorphically onto the complement of $\mathbb{CP}^1$, and mapping $S^3 = \partial D^4 \rightarrow \mathbb{CP}^1$ by a map whose fibers are great circles in S^3 . Generalize this to give a cell structure for $\mathbb{CP}^n$ for any $n < \infty$.

Problem 2. (Invariance of Domain) Show that if $n \neq m$ are non-negative integers and $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are non-empty open subsets, then U and V are not homeomorphic.

Problem 3. Let A and B be abelian groups. Show that $\text{Tor}(A, B)$ and $\text{Tor}(B, A)$ are naturally identified. [Hint: Let $0 \rightarrow F_2 \rightarrow F_1 \rightarrow A \rightarrow 0$ and $0 \rightarrow G_2 \rightarrow G_1 \rightarrow B \rightarrow 0$ and form a commutative square whose (i, j) -term is $F_i \otimes G_j$ (where $F_0 = A$ and $G_0 = B$), and do an extended version of the connecting homomorphism construction to prove the equivalence.]

Problem 4. Let X be a topological space and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover whose open sets are indexed by a set A . Define the Čech cochains of this cover by

$$C^p(\mathcal{U}; \mathbb{Z}) = \prod_{(\alpha_0, \dots, \alpha_p) \in A^{p+1}} C^0(U_{\alpha_0} \cap \dots \cap U_{\alpha_p}; \mathbb{Z}),$$

the product, over all $(p+1)$ -tuples of indices, of the groups $C^0(-; \mathbb{Z})$ of locally constant $\mathbb{Z}$ -valued functions on the $(p+1)$ -fold intersections of elements of $\mathcal{U}$. Define

$$\delta(\{f_\alpha\}_{\alpha \in A^{p+1}})(U_{\alpha_0} \cap \dots \cap U_{\alpha_{p+1}}) = \sum_{i=0}^{p+1} (-1)^i f_{(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1})}(U_{\alpha_0} \cap \dots \cap U_{\alpha_{p+1}}).$$

Show that this forms a cochain complex, the *Čech cochain complex* of the open covering. Compute the Čech cohomology for the covering $\{S^1 \setminus \{n\}, S^1 \setminus \{s\}\}$ of S^1 .

Problem 5. By a *cell* we mean a compact convex body with non-empty interior in some Euclidean space cut out by finitely many affine linear inequalities. Then the facets themselves are cells in one lower dimension and their facets are cells in codimension 2, etc. A regular cell complex is a space obtained by gluing compact convex cells together along faces by affine linear isomorphisms. A barycentric subdivision of such is obtained by choosing a point b_a in the interior of each cell a (ranging over the cells of all dimensions) and forming simplices by taking the convex hull of sets of points of the form $\{b_0, b_1, \dots, b_k\}$, provided that for each $i < j$ the closed cell containing b_j in its interior contains the

cell containing b_i in its interior. Show that this constructs a simplicial complex and a homeomorphism from its geometric realization to the cell complex with the property that each simplex is contained in a cell and the map from the simplex to the cell is an affine linear injection.

Problem 6. Let K be a simplicial complex. For a simplex σ of K an *orientation* of σ is determined by an ordering of its vertices. By definition, any other ordering differing from the given one by an even permutation determines the same orientation. The orderings differing by an odd permutation determine the *opposite* orientation or the *negative* of the orientation. An ordering for a simplex determines an ordering for each facet by taking the induced ordering. The formula $\partial\sigma = \sum_{i=0}^n (-1)^i \partial_i(\sigma)$ can be interpreted as giving an orientation to each face of σ from an ordering of the vertices of σ .

(a.) Show that the induced orientation on a boundary depends only on the orientation of σ , not on the ordering that determines it.

(b.) Define chain groups C_* whose k -chains are the quotient of the free abelian group generated by the oriented k simplices modulo the relation that the two orientations of a simplex give negative elements in the group. Using the discussion above to define the boundary formula for this chain complex.

(c.) Show that this is indeed a chain complex.

(d.) Show that the chain complex is free abelian and the rank of C_k is the number of k -simplices of K .

(e.) Compute the homology of this chain complex for $\partial\Delta^n$.