

PROBLEM SET 1

Problem 1. For topological spaces X, Y , let $\text{Map}(X, Y)$ denote the space of continuous maps with the *compact-open topology*. [The sub-basic open subsets of this topology, denoted $\langle K, U \rangle$ with $K \subset X$ compact and $U \subset Y$ open, are the sets of maps f with the property that $f(K) \subset U$.]

- (a) Show that a homotopy $h: X \times I \rightarrow Y$ defines a path in $\text{Map}(X, Y)$ by $t \mapsto h_t: X \rightarrow Y$.
 (b) Show that if X is a locally compact Hausdorff space, then conversely, for a path $\omega: I \rightarrow \text{Map}(X, Y)$, the map $h: X \times I \rightarrow Y$ given by $h(x, t) = \omega(t)(x)$ is a homotopy. For general X the notions do not agree, but the best notion to work with is homotopy, rather than continuous paths in $\text{Map}(X, Y)$.

Problem 2. (a) Suppose we have a map $\varphi: \Delta^n \rightarrow X$ and a homotopy $H: \partial\Delta^n \times I \rightarrow X$ with $H(x, 0) = \varphi(x)$ for all $x \in \partial\Delta^n$. Show that there is $\hat{H}: \Delta^n \times I \rightarrow X$ with $\hat{H}|_{\partial\Delta^n \times I} = H$ and $\hat{H}(x, 0) = \varphi(x)$ for all $x \in \Delta^n$.

- (b) Show that if K is a finite simplicial complex, $L \subseteq K$ a subcomplex, and $\varphi: |K| \rightarrow X$ and $H: |L| \times I \rightarrow X$ are maps with $H(y, 0) = \varphi(y)$ for all $y \in |L|$, then there is a homotopy $\hat{H}: |K| \times I \rightarrow X$ with $\hat{H}(x, 0) = \varphi(x)$ for all $x \in |K|$ and $\hat{H}(y, t) = H(y, t)$ for all $y \in |L|$.

Problem 3. (a) Show that a contractible space has trivial fundamental group. [Warning: You cannot assume that the base point is fixed during the contraction.]

- (b) Show that every map $\varphi: D^2 \rightarrow D^2$ has a fixed point.

Problem 4. Compute the fundamental group of the following spaces: A wedge of k circles; S^2 ; S^n for $n > 2$, $\mathbb{R}P^2$, the real projective plane, which is the quotient of S^2 by the antipodal involution; $\mathbb{R}P^n$, the n -dimensional real projective space; the 2-torus $S^1 \times S^1$; a closed oriented surface of genus $g > 1$.

Problem 5. (a) Let (X, A) be a pair of spaces and let $x_0 \in A$ be a basepoint. Show that there is a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(X, A, x_0) \rightarrow \pi_n(A, x_0) \rightarrow \pi_n(X, x_0) \rightarrow \pi_n(X, A, x_0) \longrightarrow \cdots$$

ending with $\pi_1(X, A, x_0) \rightarrow \pi_0(A, x_0) \rightarrow \pi_0(X, x_0)$. All the terms are groups except $\pi_1(X, A, x_0)$ and the two sets π_0 , which carry only a distinguished element; exactness is meant throughout in the sense that the image of each map is the set of elements sent by the next map to the distinguished element.

- (b) Show that $\pi_{n+1}(D^k, S^{k-1}, x_0) \cong \pi_n(S^{k-1}, x_0)$ for $n \geq 1$.

Problem 6. (a) Show by induction on k that the k -sphere can be triangulated with simplices of arbitrarily small diameter in the usual spherical metric.

- (b) Let $(x^1, \dots, x^{n+1})$ be the usual coordinates on $\mathbb{R}^{n+1}$. Decompose the unit sphere $S^n \subset \mathbb{R}^{n+1}$ as $U_+ \cup U_-$ where $U_+ = \{p \in S^n : x^{n+1}(p) > -1/2\}$ and $U_- = \{p \in S^n :$

$x^{n+1}(p) < 1/2\}$. Let $f: S^k \rightarrow S^n$ be given. Use (a) to show that there is a triangulation of the domain so that under f each simplex maps to either U_+ or U_- .

(c) Show that $\pi_k(S^n, (1, 0, \dots, 0))$ is trivial for all $k < n$.

Problem 7. Let S^3 be the unit sphere in two-dimensional complex linear space, $\mathbb{C}^2$. Using complex multiplication define an action of S^1 on S^3 . Show that it is free and that the quotient space can be identified with S^2 . This defines a map $S^3 \rightarrow S^2$, called the *Hopf map*, with every fiber a circle in S^3 (in fact a great circle — the intersection of S^3 with a complex line through the origin). We shall eventually show using the cohomology ring structure, or some elementary smooth topology, that this map is homotopically non-trivial, and in fact generates an infinite cyclic subgroup of $\pi_3(S^2, x_0)$ for any base point x_0 of S^2 .