

BASIC CONCEPTS OF TOPOLOGY: SEPARATION, COUNTABILITY AND PARACOMPACTNESS

These notes collect the point-set hypotheses that recur throughout algebraic topology. The countability axioms say that a space is describable by countably much data, either globally or near each point. The separation axioms say that the topology has enough open sets to tell points and closed sets apart. Paracompactness is the hypothesis that makes partitions of unity possible and allows one to assemble local constructions into global ones. Here, all spaces are topological spaces and all maps between them are continuous unless explicitly stated otherwise.

1. BASES AND THE COUNTABILITY AXIOMS

Definition 1.1. Let X be a topological space. A family $\mathcal{B}$ of open subsets of X is a *base* for the topology if every open set is a union of members of $\mathcal{B}$; equivalently, if for every open U and every $x \in U$ there is $B \in \mathcal{B}$ with $x \in B \subseteq U$. A *sub-basis* for the topology is a collection of open sets such that all finite intersections of the sets from the collection form a basis for the topology.

A *neighborhood* of x is a set N with $x \in U \subseteq N$ for some open U . A family $\mathcal{N}_x$ of neighborhoods of x is a *neighborhood base at x* if every neighborhood of x contains a member of $\mathcal{N}_x$.

Definition 1.2. A space X is

- (i) *first countable* if every point of X has a countable neighborhood base;
- (ii) *second countable* if the topology of X has a countable base.

The two axioms are related in one direction only.

Proposition 1.3. *Every second countable space is first countable. The converse is false.*

Proof. Let $\mathcal{B}$ be a countable base and let $x \in X$. Put $\mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$, a countable family of neighborhoods of x . If N is a neighborhood of x , choose open U with $x \in U \subseteq N$; since $\mathcal{B}$ is a base there is $B \in \mathcal{B}$ with $x \in B \subseteq U \subseteq N$, and $B \in \mathcal{B}_x$. Hence $\mathcal{B}_x$ is a countable neighborhood base at x .

For the converse, let X be an uncountable set with the discrete topology. Each point x has the one-element neighborhood base $\{\{x\}\}$, so X is first countable. But every base must contain $\{x\}$ for every x — $\{x\}$ is open, and the only subset of it containing x is itself — so no base is countable. $\square$

Definition 1.4. Let X be a space. A family $\mathcal{V}$ of subsets is a *cover* if its union is X , and an *open cover* if in addition every member is open. A cover $\mathcal{V}$ *refines* a cover $\mathcal{U}$, and is a *refinement* of it, if every $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$. A subcover is a refinement, but a refinement need not be a subcover: its members need not belong to $\mathcal{U}$ at all.

Definition 1.5. X is *separable* if it has a countable dense subset, and *Lindelöf* if every open cover of X has a countable subcover.

Proposition 1.6. *A second countable space is separable and Lindelöf, and every subspace of a second countable space is second countable.*

Proof. Let $\mathcal{B} = \{B_1, B_2, \dots\}$ be a countable base, discarding the empty set. Choosing $x_n \in B_n$ gives a countable set $D = \{x_n\}_{n=1}^\infty$ meeting every non-empty B_n . Since every non-empty open set contains some B_n , the set D meets every open set, and hence its closure is all of X . That is to say, D is dense.

Let $\mathcal{U}$ be an open cover. Let S be the set of indices n for which $B_n \subseteq U$ for at least one $U \in \mathcal{U}$, and for each $n \in S$ choose such a U_n . Given $x \in X$, pick $U \in \mathcal{U}$ containing x and then B_n with $x \in B_n \subseteq U$; then $n \in S$ and $x \in B_n \subseteq U_n$. Hence $\{U_n\}_{n \in S}$ is a countable subcover.

Finally, if $A \subseteq X$ then $\{B_n \cap A\}$ is a countable base for the subspace topology. $\square$

Example 1.7. $\mathbb{R}^n$ is second countable: the balls $B(q, r)$ with $q \in \mathbb{Q}^n$ and $r \in \mathbb{Q}_{>0}$ form a countable base. Every metric space is first countable, the balls $B(x, 1/k)$, $k \geq 1$, forming a neighborhood base at x .

Example 1.8 (the Sorgenfrey line). Let $\mathbb{R}_\ell$ be $\mathbb{R}$ with the topology generated by the half-open intervals $[a, b)$. It is first countable — the sets $[x, x + 1/k)$ are a neighborhood base at x — and separable, since $\mathbb{Q}$ is dense. It is not second countable: if $\mathcal{B}$ is a base, then for each x there is $B_x \in \mathcal{B}$ with $x \in B_x \subseteq [x, x + 1)$, and $x = \inf B_x$, so $x \mapsto B_x$ is injective and $\mathcal{B}$ is uncountable. Thus separability does not imply second countability in general.

Theorem 1.9. *For a metrizable space X the following are equivalent: X is second countable; X is separable; X is Lindelöf.*

Proof. Second countable $\Rightarrow$ separable and Lindelöf is Proposition 1.6 and needs no metric.

Separable $\Rightarrow$ second countable. Let $D = \{x_1, x_2, \dots\}$ be countable and dense and let $\mathcal{B} = \{B(x_n, 1/k) : n, k \geq 1\}$, a countable family. If U is open and $x \in U$, choose k with $B(x, 2/k) \subseteq U$ and then $x_n \in B(x, 1/k)$ by density; then $x \in B(x_n, 1/k) \subseteq B(x, 2/k) \subseteq U$ by the triangle inequality.

Lindelöf $\Rightarrow$ separable. For each k the cover $\{B(x, 1/k)\}_{x \in X}$ has a countable subcover, with centers D_k say; then $D = \bigcup_k D_k$ is countable and dense. $\square$

Proposition 1.10 (sequences suffice in first countable spaces). *Let X be first countable, $A \subseteq X$ and $f: X \rightarrow Y$.*

- (i) $x \in \overline{A}$ if and only if some sequence in A converges to x .
- (ii) f is continuous at x if and only if $f(x_n) \rightarrow f(x)$ for every sequence $x_n \rightarrow x$.

Proof. Let $N_1, N_2, \dots$ be a countable neighborhood base at x ; replacing N_k by $N_1 \cap \dots \cap N_k$ we may assume it is decreasing. For (i), if $x \in \overline{A}$ choose $a_k \in A \cap N_k$; then $a_k \rightarrow x$, since any neighborhood of x contains some N_k and hence all a_j with $j \geq k$. The converse holds in any space. For (ii), suppose f is not continuous at x : there is a neighborhood W of $f(x)$ with $f(N_k) \not\subseteq W$ for every k , so choosing $x_k \in N_k$ with $f(x_k) \notin W$ gives $x_k \rightarrow x$ while $f(x_k) \not\rightarrow f(x)$. Again the converse holds in any space. $\square$

2. SEPARATION: HAUSDORFF SPACES

Definition 2.1. A space X is

- (T_1) T_1 if every one-point subset is closed;
 (T_2) *Hausdorff* if for all $x \neq y$ in X there are disjoint open sets $U \ni x$ and $V \ni y$.

Hausdorff implies T_1 : if $x \neq y$, the open set V above shows $y \notin \overline{\{x\}}$. The converse fails, as Example 2.4 shows.

Proposition 2.2. *Let X be Hausdorff.*

- (i) *Limits of convergent sequences (indeed of nets) are unique.*
- (ii) *Every compact subset of X is closed, and disjoint compact subsets can be separated by disjoint open sets.*
- (iii) *If $f, g: Y \rightarrow X$ are continuous and agree on a dense subset of Y , then $f = g$.*

Moreover X is Hausdorff if and only if the diagonal $\Delta = \{(x, x)\}$ is closed in $X \times X$.

Proof. (i) If $x_n \rightarrow x$ and $x_n \rightarrow y$ with $x \neq y$, separate x and y by disjoint U, V ; the sequence is eventually in each, which is absurd.

(ii) Let $C \subseteq X$ be compact and $x \notin C$. For each $y \in C$ choose disjoint open $U_y \ni x$, $V_y \ni y$. Finitely many $V_{y_1}, \dots, V_{y_k}$ cover C , and $U_{y_1} \cap \dots \cap U_{y_k}$ is a neighborhood of x missing C ; so $X \setminus C$ is open. Now let C and D be disjoint compact sets. The argument just given, with C in place of the point x , provides for each $x \in D$ disjoint open sets $U_x \supseteq C$ and $V_x \ni x$; finitely many $V_{x_1}, \dots, V_{x_k}$ cover D , and then $\bigcap_i U_{x_i}$ and $\bigcup_i V_{x_i}$ are disjoint open sets containing C and D .

(iii) $\{y : f(y) = g(y)\}$ is closed, being the preimage of Δ under (f, g) and Δ being closed by the last statement, and it contains a dense set.

For the last statement, Δ is closed if and only if every (x, y) with $x \neq y$ has a basic neighborhood $U \times V$ missing Δ , i.e. with $U \cap V = \emptyset$. $\square$

Example 2.3. Every metric space is Hausdorff: if $x \neq y$ and $r = d(x, y)/2$ then $B(x, r)$ and $B(y, r)$ are disjoint.

Example 2.4 (the cofinite topology). Let X be infinite with the topology whose closed sets are X and the finite subsets. Then X is T_1 , but any two non-empty open sets meet, so X is not Hausdorff.

The next two axioms separate points from closed sets and closed sets from each other; they are what paracompactness will deliver.

Definition 2.5. A T_1 space X is *regular* if for every closed A and every $x \notin A$ there are disjoint open $U \ni x$ and $V \supseteq A$, and *normal* if for all disjoint closed A, B there are disjoint open $U \supseteq A$ and $V \supseteq B$.

With the T_1 hypothesis included, normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff. Every metric space is normal: for disjoint closed A, B take $U = \{x : d(x, A) < d(x, B)\}$ and $V = \{x : d(x, B) < d(x, A)\}$.

3. COVERS, LOCAL FINITENESS AND PARACOMPACTNESS

Definition 3.1. A space X is *locally compact* if every point of X has a compact neighborhood. In a Hausdorff space this is equivalent to the stronger-looking condition that every point has a neighborhood base of open sets with compact closure, which is the form in which it is used below.

Definition 3.2. A family $\mathcal{V}$ of subsets of X is *locally finite* if every point of X has a neighborhood meeting only finitely many members of $\mathcal{V}$.

Local finiteness is much stronger than the requirement that each point lie in only finitely many members. Two consequences of local finiteness are used constantly:

Lemma 3.3. *Let $\mathcal{V} = \{V_\alpha\}$ be locally finite. Then*

- (i) $\{\overline{V_\alpha}\}$ is locally finite, and
- (ii) $\bigcup_\alpha V_\alpha = \bigcup_\alpha \overline{V_\alpha}$; in particular a locally finite union of closed sets is closed.

Proof. (i) If N is open and $N \cap V_\alpha = \emptyset$ then, since $X \setminus N$ is closed, $N \cap \overline{V_\alpha} = \emptyset$. So an open neighborhood meeting only finitely many V_α meets only finitely many $\overline{V_\alpha}$.

(ii) The inclusion $\supseteq$ is clear. Conversely let $x \in \bigcup_\alpha \overline{V_\alpha}$ and choose an open $N \ni x$ meeting only $V_{\alpha_1}, \dots, V_{\alpha_k}$. If x were in no $\overline{V_{\alpha_i}}$, then $N \setminus (\overline{V_{\alpha_1}} \cup \dots \cup \overline{V_{\alpha_k}})$ would be a neighborhood of x missing every V_α . $\square$

Definition 3.4. A space X is *paracompact* if every open cover of X has a locally finite open refinement. Many authors, and these notes, include the Hausdorff axiom in the word: a *paracompact space* is a Hausdorff space with that covering property.

Example 3.5. A compact Hausdorff space is paracompact: a finite sub-cover is a locally finite open refinement. A discrete space is paracompact, the singletons refining every cover. More generally every metrizable space is paracompact; this is a theorem of A. H. Stone, and it is not easy.

Example 3.6 (manifolds). A locally Euclidean Hausdorff space that is second countable (a manifold, in the usual and our definition) is paracompact. One route is through the two theorems just quoted: such a space is locally compact Hausdorff, since each point has a neighborhood basis of open sets with compact closure, hence regular; being also second countable it is metrizable by Urysohn's metrization theorem, hence paracompact by Stone's theorem. For manifolds, however, none of that machinery is needed: second countability yields an exhaustion by compact sets, and from it a locally finite refinement of any open cover by coordinate balls, directly and in a few lines. That argument is carried out in the companion notes on manifolds, where it is also pushed one step further, to *smooth* partitions of unity — which is what the applications require, and which the theorems quoted here do not provide. This explains exactly why second countability is built into the definition of a manifold: it is the hypothesis that yields partitions of unity, and with them Riemannian metrics, tubular neighborhoods, and the multitude of patching arguments for local constructions that permeate the subject.

Example 3.7 (a space that is not paracompact). The long line L , obtained by inserting a copy of $(0, 1)$ between consecutive points of the first uncountable ordinal, is Hausdorff, locally Euclidean and first countable, but neither second countable nor paracompact. So “locally Euclidean and Hausdorff” does not suffice.

Proposition 3.8. *A closed subspace of a paracompact space is paracompact.*

Proof. Let $A \subseteq X$ be closed and let $\mathcal{U}$ be a cover of A by sets open in A ; write each as $U = \tilde{U} \cap A$ with $\tilde{U}$ open in X . The family of these $\tilde{U}$ together with $X \setminus A$ is an open cover of X ; take a locally finite open refinement $\mathcal{W}$ and intersect its members with

A , discarding those coming from $X \setminus A$. The result is a locally finite open refinement of $\mathcal{U}$ covering A . Finally A is Hausdorff, being a subspace of a Hausdorff space, as the definition of paracompactness used here requires. $\square$

Theorem 3.9. *Every paracompact space is normal.*

We do not prove this here; the argument runs in two steps, first separating a point from a closed set using local finiteness of a refinement, which gives regularity, and then repeating the same argument with the point replaced by a closed set. The theorem is the reason paracompactness is the standard hypothesis in gluing arguments: it produces Urysohn functions, and hence the following.

Definition 3.10. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X . A *partition of unity subordinate to $\mathcal{U}$* is a family of continuous functions $\varphi_\alpha: X \rightarrow [0, 1]$ such that

- (i) $\text{supp } \varphi_\alpha = \overline{\{x : \varphi_\alpha(x) \neq 0\}} \subseteq U_\alpha$ for every α ;
- (ii) the family $\{\text{supp } \varphi_\alpha\}$ is locally finite; and
- (iii) $\sum_\alpha \varphi_\alpha(x) = 1$ for every $x \in X$, the sum being finite at each point by (ii).

Theorem 3.11. *A Hausdorff space X is paracompact if and only if every open cover of X admits a subordinate partition of unity.*

One direction is immediate. If $\{\varphi_\alpha\}$ is a partition of unity subordinate to $\mathcal{U}$, then the open sets $\varphi_\alpha^{-1}((0, 1])$ refine $\mathcal{U}$, by (i); they cover X , by (iii); and they are locally finite, being contained in the $\text{supp } \varphi_\alpha$, by (ii). The other direction we do not prove: it runs through Theorem 3.9, a shrinking lemma producing a cover $\{V_\alpha\}$ with $\overline{V_\alpha} \subseteq U_\alpha$, and Urysohn functions for the pairs $\overline{V_\alpha} \subseteq U_\alpha$, which are then normalized by dividing by their sum. For the only case used in these notes, that of a manifold, both the covering property and the partition of unity are constructed directly in the companion notes on manifolds.

4. EXERCISES

Exercise 4.1. Prove that a countable product of second countable spaces is second countable, and that an uncountable product of copies of the two-point discrete space $\{0, 1\}$ is not first countable.

Exercise 4.2. Show that second countability, first countability and the Hausdorff property are inherited by arbitrary subspaces, whereas normality and paracompactness are inherited by closed subspaces.

Exercise 4.3. Let X be a locally compact, Hausdorff, second countable space. Prove that X has an *exhaustion by compact sets*: $X = \bigcup_n K_n$ with each K_n compact and $K_n \subseteq \text{int } K_{n+1}$. (This is stronger than σ -compactness, which asks only that X be a countable union of compact sets.) Deduce that X is paracompact.